Domination and Compensation in Finite Dimension Dynamical Systems

Larbi Afifi, El Mostafa Magri, Abdelhaq El Jai

Int. Journal of Math. Analysis, Vol. 4, 2010, no. 49, 2443 - 2457

Domination and Compensation in Finite Dimension Dynamical Systems

Larbi AFIFI

Faculty of Sciences, University Hassan II Ain Chock
B.P.5366-Maârif, Casablanca, Morocco
l.afifi@fsac.ac.ma, larbi_afifi@yahoo.fr

El Mostafa MAGRI

Faculty of Sciences, University Hassan II Ain Chock
B.P.5366-Maârif, Casablanca, Morocco
m.magri@fsac.ac.ma

Abdelhaq EL JAI

MEPS - Théorie des Systèmes, Université de Perpignan
52 Avenue Paul Alduy, Perpignan Cedex
aej@univ-perp.fr

Résumé

This work concerns the analysis of a class of linear dynamical systems. We study the possibility of comparing input operators, with respect to the output one, and we give characterization results. Various situations are examined, applications and illustrative examples are presented.
Under convenient hypothesis, we also show how to find the optimal control ensuring the compensation of a disturbance in the finite time or asymptotic cases. The relationship with the notions of controllability, stability and stabilizability is examined. Here also, applications and examples illustrating different results and situations are given.

Keywords : Dynamical systems, observation, control, domination, remediability, disturbance

1 Introduction and problem statement

In this work, we consider a class of finite dimension dynamical systems described by a linear state equation as follows :

$$(S)\begin{cases} \dot{z}(t) = Az(t) + B_1u_1(t) + B_2u_2(t)\ ;\ 0 < t < T \\ z(0) = z_0 \in \mathbb{R}^n \end{cases} \tag{1}$$

where $A \in M_n(\mathbb{R})$, $B_1 \in M_{n,p}(\mathbb{R})$, $B_2 \in M_{n,m}(\mathbb{R})$, $u_1 \in L^2(0,T;\mathbb{R}^p)$ and $u_2 \in L^2(0,T;\mathbb{R}^m)$.

The system (S) is augmented by the output equation :

$$y(t) = Cz(t)\ ;\ \ 0 < t < T \tag{2}$$

where $C \in M_{q,n}(R)$. The state of system (S) at time t, is given by :

$$z(t) = e^{At}z_0 + H_1(t)u_1 + H_2(t)u_2 \tag{3}$$

where for $t \in]0,T]$, $H_1(t)$ and $H_2(t)$ are the operators defined by

$$\begin{array}{rccc} H_1(t): & L^2(0,t;\mathbb{R}^p) & \longrightarrow & \mathbb{R}^n \\ & u_1 & \longrightarrow & \displaystyle\int_0^t e^{A(t-s)}B_1u_1(s)ds \end{array} \tag{4}$$

and

$$\begin{array}{rccc} H_2(t): & L^2(0,t;\mathbb{R}^m) & \longrightarrow & \mathbb{R}^n \\ & u_2 & \longrightarrow & \displaystyle\int_0^t e^{A(t-s)}B_2u_2(s)ds \end{array} \tag{5}$$

For $i = 1, 2$, we note $H_i = H_i(T)$. Then

$$y(T) = Ce^{AT}z_0 + CH_1u_1 + CH_2u_2 \tag{6}$$

The system (S) is excited by two input terms B_1u_1 and B_2u_2 where one, the second for example, is considered as an intentional or accidental disturbance. The other term B_1u_1 is introduced in order to compensate [1,2] at the final time T, the effect of the disturbance by bringing back the observation to the normal situation which is $Ce^{AT}z_0$. That is to say : for any $u_2 \in L^2(0,T;\mathbb{R}^m)$, there exists a control $u_1 \in L^2(0,T;\mathbb{R}^p)$ such that :

$$Ce^{A(T)}z_0 + \int_0^T Ce^{A(T-s)}B_1u_1(s)ds + \int_0^T Ce^{A(T-s)}B_2u_2(s)ds = Ce^{A(T)}z_0$$

or equivalently

$$CH_1u_1 + CH_2u_2 = 0$$

This leads to the notion of domination which consists to study the possibility of comparing the input operators B_1 and B_2, with respect to the output one C.

The domination notion is introduced and studied separately for controlled and observed distributed systems [3]. In this paper, we examine the problem of domination in connection with the compensation one.

First, we consider the finite time case. We define and we characterize the notion of domination. Sufficient conditions, applications and illustrative examples are also given. The minimum energy problem [7,8] is examined using Hilbert Uniqueness Method, such a problem can be studied as a general optimal control one. The obtained results are extended to the asymptotic case and various situations are examined. The relationship with the notions of stability and stabilizability [4,5,6] are equally studied.

2 Finite time C-domination

2.1 Definitions and characterizations

We define hereafter the notion of C-domination.

Definition 2.1
We say that B_1 dominates B_2 on $[0,T]$ with respect to C (or B_1 C-dominates B_2 on $[0,T]$), if for any $u_2 \in L^2(0,T;\mathbb{R}^m)$, there exists a control $u_1 \in L^2(0,T;\mathbb{R}^p)$ such that :

$$CH_1u_1 + CH_2u_2 = 0$$

In this case, and for C and T fixed, one can note $B_2 \leq B_1$

We have the following characterization result.

Proposition 2.2 *The following properties are equivalent :*

i) B_1 dominates B_2 on $[0,T]$ with respect to C.

ii) $Im(CH_2) \subset Im(CH_1)$.

iii) $Ker(H_1^*C^*) \subset Ker(H_2^*C^*)$.

iv) $\exists \gamma > 0$ *such that for any* $\theta \in \mathbb{R}^q$*, we have*

$$\| B_2^* e^{A^*(T-.)} C^* \theta \|_{L^2(0,T;\mathbb{R}^m)} \leq \gamma \, \| B_1^* e^{A^*(T-.)} C^* \theta \|_{L^2(0,T;\mathbb{R}^p)} \tag{7}$$

Proof : Derive from the definition, the fact that

$$Ker(H_i^* C^*) = Ker(B_i^* e^{A^*(T-.)} C^*), \quad for \ i = 1, 2$$

and also the following well known result.

Lemma 2.3
Let X , Y *and* Z *be Banach reflexive spaces,* $P \in \mathcal{L}(X, Z)$ *and* $Q \in \mathcal{L}(Y, Z)$. *We have*

$$Im(P) \subset Im(Q)$$

if and only if

$$\exists \gamma > 0 \ such \ that \ for \ any \ z^* \in Z', \ we \ have \ \|P^* z^*\|_{X'} \leq \gamma \, \|Q^* z^*\|_{Y'}$$

Concerning the relationship with the controllability notion, we have the following result.

Proposition 2.4

i) If the system

$$(S_1) \begin{cases} \dot{z}(t) = Az(t) + B_1 u_1(t) \ ; \ 0 < t < T \\ z(0) = z_0 \in \mathbb{R}^n \end{cases} \tag{8}$$

is controllable on $[0, T]$*, then* B_1 *dominates any operator* B_2 *on* $[0, T]$*, with respect*[1] *to* C.

ii) The converse is not true.

Proof :

i) Obviously, (S_1) is controllable on $[0, T] \Longleftrightarrow Im H_1 = \mathbb{R}^n$,
then

$$Im\,(CH_1) = Im(C)$$

and hence

[1]In fact, this is true for any output operator C.

$$Im\,(CH_2) \subset Im\,(CH_1)$$

Consequently B_1 dominates B_2 on $[0,T]$, with respect to C.

ii) <u>Counter example</u> : We consider the case where $n = 2$, $p = q = 1$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

we have

$$B_2^* e^{A^*(T-s)} C^* \theta = e^{(T-s)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \theta = \begin{pmatrix} e^{(T-s)}\theta \\ 0 \end{pmatrix}$$

and

$$B_1^* e^{A^*(T-s)} C^* \theta = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} e^{(T-s)}\theta \\ 0 \end{pmatrix} = e^{(T-s)}\theta$$

then

$$\| B_2^* e^{A^*(T-.)} C^* \theta \|_{L^2(0,T;\mathbb{R}^2)} \quad = \quad \| B_1^* e^{A^*(T-.)} C^* \theta \|_{L^2(0,T;\mathbb{R})}$$

The inequality (7) is then true for $\gamma = 1$. Hence B_1 C-dominates B_2, but

$$rank \begin{pmatrix} B_1 & AB_1 \end{pmatrix} = rank \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 1 < 2$$

Consequently (S_1) is not controllable on $[0,T]$. □

We give hereafter a sufficient condition ensuring such a domination.

Proposition 2.5 *If*

$$rank \begin{pmatrix} CB_1 & CAB_1 & ... & CA^{n-1}B_1 \end{pmatrix} = q \tag{9}$$

then B_1 C-dominates any operator B_2 on $[0,T]$.

<u>Proof</u> : Using Caylay-Hamilton theorem, we have

$$rank \begin{pmatrix} CB_1 & CAB_1 & ... & CA^{n-1}B_1 \end{pmatrix} = q$$

$$\Longleftrightarrow \begin{pmatrix} B_1^*C^* \\ B_1^*A^*C^* \\ \vdots \\ B_1^*(A^*)^{n-1}C^* \end{pmatrix}_{(np,q)} y = 0\,; \forall y \in \mathbb{R}^q \quad \Longrightarrow y = 0$$

$$\Longleftrightarrow Ker(H_1)^*C^* = \{0\}$$

Hence, if $Ker\,[(H_1)^*C^*] = \{0\}$, then $Ker\,[(H_1)^*C^*] \subset Ker\,[(H_2)^*C^*]$ and then, B_1 dominates B_2 on $[0,T]$ with respect to C . □

Remark 2.6

i) One can have

$$rank\left(\begin{array}{cccc} CB_1 & CAB_1 & \ldots & CA^{n-1}B_1 \end{array}\right) = q$$

even if the system (S_1) *is not controllable on* $[0,T]$.

ii) B_1 *may dominates another operator* B_2, *with respect to* C *on* $[0,T]$, *without having*

$$rank\left(\begin{array}{cccc} CB_1 & CAB_1 & \ldots & CA^{n-1}B_1 \end{array}\right) = q$$

This is illustrated in the following example.

Example 2.7

i) We consider the case where $n = 2$, $p = q = 1$ *and*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

The controllability matrix is given by

$$\begin{pmatrix} B_1 & AB_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

its rank is then $1 < 2$. *Consequently, the corresponding system is not controllable on* $[0,T]$. *On the other hand*

$$\begin{pmatrix} CB_1 & CAB_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Its rank is $1 = q$, *then* B_1 *dominates any operator* B_2 *on* $[0,T]$ *with respect to* C .

ii) Now, for $m = n = 2$, $p = 1$, $q = 2$ *and*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

we have

$$e^{A^*(T-.s)}C^*\theta \;=\; e^{(T-s)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$= \; e^{(T-s)}\begin{pmatrix} \theta_1+\theta_2 \\ \theta_1+\theta_2 \end{pmatrix}$$

and $B_1^*e^{A^*(T-s)}C^*\theta = e^{(T-s)}2(\theta_1+\theta_2)$*, then*

$$\| B_1^*e^{A^*(T-.)}C^*\theta\|^2_{L^2(0,T;R)} \;=\; 4\int_0^T e^{2(T-s)}(\theta_1+\theta_2)^2 ds$$

On the other hand

$$\| B_2^*e^{A^*(T-.)}C^*\theta\|^2_{L^2(0,T;\mathbb{R}^2)} \;=\; \int_0^T e^{2(T-s)} \| \begin{pmatrix} \theta_1+\theta_2 \\ \theta_1+\theta_2 \end{pmatrix} \|^2 \, ds$$

$$= \; 2\int_0^T e^{2(T-s)}(\theta_1+\theta_2)^2 ds$$

hence

$$\| B_2^*e^{A^*(T-.)}C^*\theta\|_{L^2(0,T;\mathbb{R}^2)} \;\leq\; \| B_1^*e^{A^*(T-.)}C^*\theta\|_{L^2(0,T;\mathbb{R})}$$

Consequently, B_1 C*-dominates* B_2 *on* $[0,T]$*, even if*

$$rank\begin{pmatrix} CB_1 & CAB_1 \end{pmatrix} = rank\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 1 \neq 2$$

□

In the following result, we give a necessary and sufficient condition for such a domination.

Proposition 2.8

B_1 C*-dominates* B_2 *on* $[0,T]$*, if and only if*

$$Im\begin{pmatrix} CB_2 & CAB_2 & ... & CA^{n-1}B_2 \end{pmatrix} \subset Im\begin{pmatrix} CB_1 & CAB_1 & ... & CA^{n-1}B_1 \end{pmatrix}$$

<u>Proof</u> : Using proposition 2.2, B_1 C-dominates B_2 on $[0,T]$, if and only if

$$Ker(H_1^*C^*) \subset Ker(H_2^*C^*)$$

Using Caylay-Hamilton theorem, we deduce that for $i = 1,2$

$$y \in Ker[(H_i)^* C^*] \Longleftrightarrow \begin{pmatrix} B_i^* C^* \\ B_i^* A^* C^* \\ \vdots \\ B_i^* (A^*)^{n-1} C^* \end{pmatrix}_{(np,q)} y = 0$$

hence

$$Ker \begin{pmatrix} B_i^* C^* \\ B_i^* A^* C^* \\ \vdots \\ B_i^* (A^*)^{n-1} C^* \end{pmatrix} = Ker(H_i^* C^*)$$

Consequently, B_1 C-dominates B_2 on $[0, T]$, if and only if

$$Im \begin{pmatrix} CB_2 & CAB_2 & \ldots & CA^{n-1}B_2 \end{pmatrix} \subset Im \begin{pmatrix} CB_1 & CAB_1 & \ldots & CA^{n-1}B_1 \end{pmatrix}$$

□

Remark 2.9

1. *This necessary and sufficient condition is not depending on the time parameter T.*
2. *In the particular case where B_2 is invertible, B_1 C-dominates B_2 on $[0, T]$ if and only if*

$$rank \begin{pmatrix} CB_1 & CAB_1 & \ldots & CA^{n-1}B_1 \end{pmatrix} = rank \begin{pmatrix} C \end{pmatrix}$$

2.2 Minimum energy problem

In this part, we assume that B_1 C-dominates B_2 on $[0, T]$, then for any $v \in L^2(0, T; \mathbb{R}^m)$, there exists a control $u \in L^2(0, T; \mathbb{R}^p)$ such that

$$CH_1 u + CH_2 v = 0 \tag{10}$$

For $v \in L^2(0, T; \mathbb{R}^m)$, we examine the existence and the uniqueness of the optimal control $u \in L^2(0, T; \mathbb{R}^p)$ satisfying (10), i.e. ensuring the compensation of the opposing term $B_2 v$.

For this, we use an extension of the Hilbert Uniqueness Method. Indeed, for $\theta \in \mathbb{R}^q$, let us note :

$$\| \theta\|_* = (\int_0^T \|(H_1)^* C^*\theta\|^2_{\mathbb{R}^p} ds)^{\frac{1}{2}} = (\int_0^T \|B_1^* e^{A^*(T-s)} C^*\theta\|^2_{\mathbb{R}^p} ds)^{\frac{1}{2}}$$

$\| \theta\|_*$ is a semi-norm on $\mathbb{R}^q$.

We assume that $\| .\|_*$ is a norm on $\mathbb{R}^q$. If $Ker\,[(H_1)^*C^*] = \{0\}$, this is equivalent to the asymptotic remediability [1,2,3] of the system (1)+(2) on $[0,T]$. The corresponding inner product is given by :

$$< \theta, \sigma >_* = \int_0^T < B_1^* e^{A^*(T-s)} C^*\theta, B_1^* e^{A^*(T-s)} C^*\sigma > ds$$

and the operator $\Lambda_C : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ defined by

$$\Lambda_C\theta = CH_1(H_1)^*C^*\theta = \int_0^T Ce^{(T-s)} B_1 B_1^* e^{A^*(T-s)} C^*\theta ds$$

is symmetric and positive definite, and then invertible. We give hereafter the expression of the optimal control ensuring the compensation of the effect of the term $B_2 v$, at the final time T.

Proposition 2.10

For $v \in L^2(0,T;\mathbb{R}^m)$, there exists a unique $\theta_v \in \mathbb{R}^q$ such that

$$\Lambda_C\theta_v = -CH_2 v$$

and the control

$$u_{\theta_v}(.) = B_1^* e^{A^*(T-.)} C^*\theta_v$$

verifies

$$CH_1 u_{\theta_v} + CH_2 v = 0$$

Moreover, it is optimal and

$$\|u_{\theta_v}\|_{L^2(0,T;\mathbb{R}^p)} = \|\theta_v\|_*$$

Let us note that one can also consider a general optimal control problem with a cost function defined on $L^2(0,T;\mathbb{R}^p)$ by

$$\begin{aligned} J(u) &= < P(CH_1u + CH_2v), CH_1u + CH_2v > \\ &+ \int_0^T < Q(CH_1(t)tu + CH_2(t)v), CH_1(t)u + CH_2(t)v > dt \qquad (11) \\ &+ \int_0^T < Ru(t), u(t) > dt \end{aligned}$$

where P, Q and R are symmetric matrixes with Q positive, P and R are positive definite.

In the next section, we present an extension to the asymptotic case.

3 Extension to the asymptotic case

In this part, we consider a class of linear dynamical systems described by the following state equation

$$(S_1^\infty) \begin{cases} \dot{z}(t) = Az(t) + B_1u_1(t) + B_2u_2(t) \ ; \ t > 0 \\ z(0) = z_0 \in \mathbb{R}^n \end{cases} \qquad (12)$$

with $A \in M_n(\mathbb{R})$, $B_1 \in M_{n,p}(\mathbb{R})$, $B_2 \in M_{m,n}(\mathbb{R})$, $u_1 \in L^2(0, +\infty; \mathbb{R}^p)$ and $u_2 \in L^2(0, +\infty; \mathbb{R}^m)$

The system (12) is augmented by the output equation :

$$y(t) = Cz(t); \qquad t > 0 \qquad (13)$$

with $C \in M_{q,n}(R)$. We have

$$z(t) = e^{At}z_0 + H_1(t)u_1 + H_2(t)u_2$$

Let

$$z = \begin{pmatrix} z_+ \\ z_- \end{pmatrix}$$

where z_+ and z_- are respectively the projections of the state z on the unstable and the stable subspaces :

$$E_+ = \bigoplus_{R(e)(\lambda) \geq 0} Ker(A - \lambda I_n)^{m(\lambda)} \qquad (14)$$

and

$$E_- = \bigoplus_{R(e)(\lambda)<0} Ker(A - \lambda I_n)^{m(\lambda)} \tag{15}$$

where $m(\lambda)$ is the multiplicity of the eigenvalue λ. E_+ and E_- are invariant with respect to the operator A. We have

$$\begin{cases} (S^+) & \dot{z}^+(t) & = & A_+ z^+(t) + PBu_1(t) + PBu_2(t) \\ & & & \\ (S^-) & \dot{z}^-(t) & = & A_- z^-(t) + (I-P)Bu_1(t) + (I-P)Bu_2(t) \end{cases} \tag{16}$$

P is the projection operator on the unstable part and A_+, respectively A_-, is the matrix induced by A on E_+, respectively E_-.

In the case where we observe only the stable part, i.e. if

$$E_+ \subset \mathrm{Ker}(C)$$

the following operators $K^\infty(C)$ and $L^\infty(C)$ respectively given by

$$\begin{array}{rrcl} K^\infty(C): & L^2(0,+\infty;\mathbb{R}^p) & \longrightarrow & \mathbb{R}^q \\ & u & \longrightarrow & \displaystyle\int_0^\infty Ce^{At}B_1u(t)dt \end{array}$$

and

$$\begin{array}{rrcl} L^\infty(C): & L^2(0,+\infty;\mathbb{R}^m) & \longrightarrow & \mathbb{R}^q \\ & v & \longrightarrow & \displaystyle\int_0^\infty Ce^{At}B_2v(t)dt \end{array}$$

are well defined. We have the same result if the considered system is exponentially stable, i.e. the matrix A is such that $Re(\lambda_i) < 0$ for $i = 1, n$; where λ_1, ..., λ_n are the eigenvalues of A. But as it will be shown later, this is not necessary.

We assume that operators $K^\infty(C)$ and $L^\infty(C)$ are well defined. By the same, we say that B_1 C-dominates B_2 asymptotically, if for every $u_2 \in L^2(0,+\infty;\mathbb{R}^m)$, there exists $u_1 \in L^2(0,+\infty;\mathbb{R}^p)$ such that :

$$K^\infty(C)u_1 + L^\infty(C)u_2 = 0$$

With a similar approach, it is easy to show the following characterization result of the asymptotic domination.

Proposition 3.1 *The following properties are equivalent :*

i) B_1 C*-dominates* B_2 *asymptotically.*

ii) $Im\,[L^\infty(C)] \subset Im\,[K^\infty(C)]$

iii) $Ker\,[K^\infty(C)] \subset Ker\,[L^\infty(C)]$

iv) $\exists \gamma > 0$ *such that* $\forall \theta \in \mathbb{R}^q$, *we have*

$$\| B_2^* e^{A^* \cdot} C^* \theta \|_{L^2(0,+\infty;\mathbb{R}^m)} \leq \gamma \| B_1^* e^{A^* \cdot} C^* \theta \|_{L^2(0,+\infty;\mathbb{R}^p)}$$

v)

$$Im \begin{pmatrix} CB_2 & CAB_2 & ... & CA^{n-1}B_2 \end{pmatrix} \subset Im \begin{pmatrix} CB_1 & CAB_1 & ... & CA^{n-1}B_1 \end{pmatrix}$$

Let us note that if the eigenvalues $\lambda_1, ..., \lambda_n$ of A are such that : $Re(\lambda_i) < 0$ for $i = 1, n$, the proposition 2.4 remain true in the asymptotic case.
One can also consider the asymptotic optimal control problem. The approach and the results are similar.

We give hereafter illustrative examples showing particularly that the notions of stability (or even the stabilizability) and also the controllability are not necessary for considering the asymptotic domination.

Example 3.2

Let us consider the case of an unstable system with $m = n = 2$, $p = q = 1$ *and*

$$A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & -3 \end{pmatrix}$$

Obviously, the system is not stable. We have

$$e^{tA} = P e^{t\Delta} P^{-1}$$

where

$$\Delta = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}; \quad P = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$$

then

$$Ce^{tA} = \begin{pmatrix} e^{-t} & -3e^{-t} \end{pmatrix}$$

In this case $K^{\infty}(C)$ *and* $L^{\infty}(C)$ *are well defined and*

$$\begin{pmatrix} CB_1 & CAB_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

and rank $\begin{pmatrix} C \end{pmatrix} = 1$. *Consequently,* B_1 C*-dominates* B_2 *asymptotically.*

□

Example 3.3 *We consider an unstable system with* $m = n = 2$, $p = q = 1$ *and* B_2 *is invertible.*

i) For

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}; \quad F = \begin{pmatrix} a & b \end{pmatrix}$$

we have

$$A + B_1 F = \begin{pmatrix} 1+a & b \\ 0 & 2 \end{pmatrix}$$

(A, B_1) *is not stabilizable because for any* $F = \begin{pmatrix} a & b \end{pmatrix}$, $A + B_1 F$ *is not stable. However,* B_1 C*-dominates* B_2 *asymptotically.*

ii) Let

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

For $F = \begin{pmatrix} a & b \end{pmatrix}$, *the matrix* $A + B_1 F = \begin{pmatrix} -1 & 0 \\ a+1 & b+2 \end{pmatrix}$ *is stable for* $b < -2$, *then* (A, B_1) *is stabilizable.*

On the other hand, we have $AB_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, *then rank*$(B_1 \ AB_1) = 1 \neq 2$ *and consequently* (A, B_1) *is not controllable (in fact, the asymptotic controllability is not well defined in the considered case). But* B_1 C*-dominates* B_2 *asymptotically, because*

$$\mathit{rank} \begin{pmatrix} CB_1 & CAB_1 \end{pmatrix} = \mathit{rank} \begin{pmatrix} C \end{pmatrix} \tag{17}$$

Let us remark that in the general case, if C *is an invertible matrix (for example if* C *is the identity matrix), then the relation (9) (the relation (17) in the considered example) is equivalent to the controllability rank condition. But this hypothesis is strong and is not very useful.*

□

Finally let us note that concerning the asymptotic minimum energy problem, and with an extension of H.U.M., it is easy to show the existence and the uniqueness of the optimal control and also how to find it.

With convenient hypothesis, one can also consider the asymptotic version of the cost function given by (11).

Acknowledgment : The authors wish to thank the Academy Hassan II of Sciences and Technics for its support to the Systems Theory Network.

Références

[1] L. Afifi, A. Chafiai and A. El Jai (2001)
Spatial Compensation of boundary disturbances by boundary actuators the International Journal of Applied Mathematics and Computer Science, Vol. 11, N°4.

[2] L. Afifi, A. El Jai and E. M. Magri (2008)
Compensation problem in finite dimension linear dynamical systems. International Journal of Applied Mathematical Sciences. Vol.2, N.45, pp 2219 - 2228.

[3] L. Afifi, M. Bahadi, A. El Jai et A. El Mizane (2009) *Asymptotic Analysis, Approximtions and Simulations of the Compensation Problem in Hyperbolic systems.* Applied Mathematical Sciences, volume 3, N°15, 737-765.

[4] L. Afifi, E. M. Magri and A. El Jai (2010)
Weak and exact domination in distributed systems. To appear in the International Journal of Applied Mathematics and Computer Science. Vol. 20, No. 3.

[5] R.F. Curtain and A.J. Pritchard (1978)
Infinite Dimensional Linear Systems Theory. Lecture Notes in Control and Information Sciences, vol. 8, Berlin.

[6] Curtain, R.F. and Zwart, H.J. (1995). *An Introduction to Infinite-Dimensional Linear Systems Theory.* Texts in Applied Mathematics, Springer-Verlag, New York.

[7] J.-L. Lions (1980)
On the foundation of optimal control of distributed systems. Lecture notes in control and inf. sc. 22, Springer.

[8] J.-L. Lions (1988)
Contrôlabilité exacte, perturbations et stabilisation des systèmes distribués. Paris, Masson.

Received : July, 2010