

An Evolutionary Boundary Value Problem

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Abstract. We consider a nonlinear initial boundary value problem in a two-dimensional rectangle. We derive variational formulation of the problem which is in the form of an evolutionary variational inequality in a product Hilbert space. Then, we establish the existence of a unique weak solution to the problem and prove the continuous dependence of the solution with respect to some parameters. Finally, we consider a second variational formulation of the problem, the so-called dual variational formulation, which is in a form of a history-dependent inequality associated with a time-dependent convex set. We study the link between the two variational formulations and establish existence, uniqueness, and equivalence results.

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1. Introduction

Our aim in this paper is to provide the variational analysis of an initial boundary value problem using arguments of evolutionary variational inequalities and history-dependent operators. The theory of variational inequalities started in early 60, based on the arguments of monotonicity and convexity. Classical references in the mathematical and numerical analysis of variational inequalities are [1, 2, 4, 5, 11, 13, 14], for instance. Various applications in Mechanics and, more specifically, in Contact Mechanics could be found in the books [3, 6, 10–12, 16–18, 21] and in the special issue [15]. Evolutionary variational inequalities are inequalities which involve the time derivative of the solution and, therefore, they require an initial condition. Existence and uniqueness results for such inequalities can be found in the books [6, 9, 19, 21], for instance. Recently, there is an interest in the study of a special class of inequalities, the so-called history-dependent variational inequalities. There are inequalities in which various functions or operators depend on the history of the solution. Their study is motivated by important application in problems involving constitutive laws for materials with memory, total slip, or total slip

rate friction laws. Existence, uniqueness, and regularity uniqueness results for such kind of inequalities can be found in [20–22], for instance.

The problem we are interested in this paper leads, in a primal variational formulation, to an evolutionary variational inequality. In contrast, its dual variational formulation is in a form of a history-dependent variational inequality. To introduce this problem, let L , h , and T be given positive constants and denote $\Omega = (0, L) \times (-h, h)$. Everywhere below we use the notation (x, y) for a generic point in Ω , and the subscripts x and y represent the partial derivative with respect to these variables. In addition, we denote by $t \in [0, T]$ the time variable and the dot above will represent the derivative with respect to t . The problem under consideration is the following.

Problem. \mathcal{P} . Find the functions $u = u(x, y, t) : [0, L] \times [-h, h] \times [0, T] \rightarrow \mathbb{R}$ and $w = w(x, t) : [0, L] \times [0, T] \rightarrow \mathbb{R}$, such that

$$\lambda \dot{u}_{xx} + Eu_{xx} + \mu \dot{u}_{yy} + Gu_{yy} + q_B = 0 \quad (1.1)$$

$$\text{for all } (x, y) \in \Omega, t \in [0, T],$$

$$\mu \dot{w}_{xx} + Gw_{xx} + (\lambda - \mu) \dot{u}_{xy} + (E - G)u_{xy} + f_B = 0 \quad (1.2)$$

$$\text{for all } (x, y) \in \Omega, t \in [0, T],$$

$$u(0, y, t) = w(0, t) = 0 \quad \text{for all } y \in [-h, h], t \in [0, T], \quad (1.3)$$

$$\lambda \dot{u}_x(L, y, t) + Eu_x(L, y, t) = 0 \quad (1.4)$$

$$\text{for all } y \in [-h, h], t \in [0, T],$$

$$\mu(\dot{u}_y(L, y, t) + \dot{w}_x(L, y, t)) + G(u_y(L, y, t) + w_x(L, y, t)) = 0 \quad (1.5)$$

$$\text{for all } y \in [-h, h], t \in [0, T].$$

$$\mu(\dot{u}_y(x, h, t) + \dot{w}_x(x, t)) + G(u_y(x, h, t) + w_x(x, t)) = q_N(x, t) \quad (1.6)$$

$$\text{for all } x \in [0, L], t \in [0, T],$$

$$(\lambda - 2\mu) \dot{u}_x(x, h, t) + (E - 2G)u_x(x, h, t) = f_N(x, t) \quad (1.7)$$

$$\text{for all } x \in [0, L], t \in [0, T].$$

$$|(\lambda - 2\mu)(\dot{u}_x(x, -h, t) + (E - 2G)u_x(x, -h, t))| \leq g, \quad (1.8)$$

$$-(\lambda - 2\mu)(\dot{u}_x(x, -h, t) - (E - 2G)u_x(x, -h, t)) = g \frac{\dot{w}(x, t)}{|\dot{w}(x, t)|}$$

$$\text{if } \dot{w}(x, t) \neq 0, \quad \text{for all } x \in [0, L], t \in [0, T],$$

$$\mu(\dot{u}(x, -h, t) + \dot{w}(x, t)) + G(u_y(x, -h, t) + w_x(x, t)) = 0 \quad (1.9)$$

$$\text{for all } x \in [0, L], t \in [0, T],$$

$$u(x, y, 0) = u_0(x, y), \quad w(x, 0) = w_0(x), \quad (1.10)$$

$$\text{for all } x \in [0, L], y \in [-h, h].$$

Problem \mathcal{P} describes the equilibrium of a viscoelastic plate submitted to the action of body forces and tractions and to nonlinear contact conditions on part of its boundary. Here, Ω represents the cross section of the plate, u is the horizontal displacement, and w is the vertical displacement. The constants λ and μ are positive viscosity coefficients, and E and G are positive

elastic coefficients. A brief description of equations and boundary condition in Problem \mathcal{P} , including their mechanical significance, is as follows.

First, Eqs. (1.1) and (1.2) represent the equilibrium equation in which the functions $q_B = q_B(x, y, t) : \Omega \times [0, T] \rightarrow \mathbb{R}$ and $f_B = f_B(x, y, t) : \Omega \times [0, T] \rightarrow \mathbb{R}$ are the horizontal and vertical components of the body forces. Condition (1.3) shows that the plate is fixed on the boundary $x = 0$, and conditions (1.4) and (1.5) show that the boundary $x = L$ is free of tractions. Next, conditions (1.6) and (1.7) represent the traction conditions. Here, the functions $q_N = q_N(x, t) : [0, L] \times [0, T] \rightarrow \mathbb{R}$ and $f_N = f_N(x, t) : [0, L] \times [0, T] \rightarrow \mathbb{R}$ denote the horizontal and vertical components of the traction forces which act on the top $y = h$ of the plate. Condition (1.8) represents the multivalued contact condition on the bottom $x = -h$ in which $g \geq 0$ is given. Condition (1.9) represents the frictionless condition, and finally, (1.10) represents the initial condition, in which the functions u_0 and w_0 are the initial horizontal and vertical displacement, respectively.

The rest of paper is structured as follows. In Sect. 2, we list the assumptions on the data and derive the variational formulation of problem \mathcal{P} . In Sect. 3, we state and prove our main result, Theorem 3.1, which states the unique weak solvability of the problem. The proof is based on arguments of evolutionary variational inequalities. In Sect. 4, we state and prove a convergence result, Theorem 4.1. It states the continuous dependence of the solution with respect to the data. Finally, in Sect. 5, we introduce the dual variational formulation of Problem \mathcal{P} for which we prove an equivalence result, Theorem 5.2.

2. Variational Formulation

We start with some notation and preliminaries. Given a real Hilbert space Y , we denote by $\langle \cdot, \cdot \rangle_Y$ its inner product and by $\| \cdot \|_Y$ the associate norm, i.e., $\|y\|_Y^2 = \langle y, y \rangle_Y$ for all $y \in Y$. For a normed space Y , we denote by $C([0, T]; Y)$ the space of the continuous functions defined on $[0, T]$ with values to Y , equipped with the canonic norm. Moreover, $\| \cdot \|_{\mathcal{L}(Y, Z)}$ denotes the norm in the space of linear continuous operators on Y with values on the normed space Z .

Everywhere below we use the standard notation for Lebesgue and Sobolev spaces. In addition, recalling that $\Omega = (0, L) \times (-h, h)$, we introduce the spaces

$$V = \{u \in H^1(\Omega) : u(0, \cdot) = 0\}, \quad W = \{w \in H^1(0, L) : w(0) = 0\}. \quad (2.1)$$

Note that equalities $u(0, \cdot) = 0$ and $w(0) = 0$ in the definitions of the spaces V and W are understood in the sense of traces. The spaces V and W are real Hilbert spaces with the canonical inner products defined by

$$\langle u, \psi \rangle_V = \iint_{\Omega} (u\psi + u_x\psi_x + u_y\psi_y) \, dx dy \quad \forall u, \psi \in V, \quad (2.2)$$

$$\langle w, \varphi \rangle_W = \int_0^L (w\varphi + w_x\varphi_x) \, dx \quad \forall w, \varphi \in W. \quad (2.3)$$

We also consider the product space $X = V \times W$ equipped with the canonical inner product given by

$$\langle \mathbf{u}, \mathbf{v} \rangle_X = \langle u, \psi \rangle_V + \langle w, \varphi \rangle_W \quad \forall \mathbf{u} = (u, w), \mathbf{v} = (\psi, \varphi) \in X. \quad (2.4)$$

On the data of Problem \mathcal{P} , we make the following hypothesis:

$$\lambda > 0, \quad E > 0, \quad \mu > 0, \quad G > 0. \quad (2.5)$$

$$f_B \in C([0, T]; L^2(\Omega)), \quad q_B \in C([0, T]; L^2(\Omega)). \quad (2.6)$$

$$f_N \in C([0, T]; L^2(0, L)), \quad q_N \in C([0, T]; L^2(0, L)). \quad (2.7)$$

$$g \geq 0. \quad (2.8)$$

$$u_0 \in V, \quad w_0 \in W. \quad (2.9)$$

Under these assumptions, we define the operators $A, B : X \rightarrow X$, functional $j : X \rightarrow \mathbb{R}$, and function $\mathbf{f} : [0, T] \rightarrow X$ by equalities

$$\langle A\mathbf{u}, \mathbf{v} \rangle_X = \lambda \iint_{\Omega} u_x \psi_x \, dx dy + \mu \iint_{\Omega} (u_y + w_x)(\psi_y + \varphi_x) \, dx dy, \quad (2.10)$$

$$\langle B\mathbf{u}, \mathbf{v} \rangle_X = E \iint_{\Omega} u_x \psi_x \, dx dy + G \iint_{\Omega} (u_y + w_x)(\psi_y + \varphi_x) \, dx dy, \quad (2.11)$$

$$j(\mathbf{v}) = g \int_0^L |\varphi| \, dx, \quad (2.12)$$

$$\begin{aligned} \langle \mathbf{f}(t), \mathbf{v} \rangle_X \\ = \iint_{\Omega} q_B(t) \psi \, dx dy + \iint_{\Omega} f_B(t) \varphi \, dx dy + \int_0^L q_N(t) \psi \, dx + \int_0^L f_N(t) \varphi \, dx, \end{aligned} \quad (2.13)$$

for all $\mathbf{u} = (u, w), \mathbf{v} = (\psi, \varphi) \in X, t \in [0, T]$. We also consider the initial data $\mathbf{u}_0 \in X$ given by

$$\mathbf{u}_0 = (u_0, v_0). \quad (2.14)$$

Note that the definitions above we do not specify the dependence of various functions on the variables x and y .

The variational formulation of Problems \mathcal{P} follows from a tedious calculus, based on the standard arguments. For this reason, we skip the details and we restrict ourselves to describe the main steps of this calculus. We proceed formally. Thus, we assume in what follows that $\mathbf{u} = (u(x, y, t), w(x, t))$ represents a regular solution to the problem \mathcal{P} , $\mathbf{v} = (\psi(x, y), \varphi(x))$ is an arbitrary element of X and $t \in [0, T]$ is fixed. Then, multiplying (1.1) by $\psi - \dot{u}$, integrating the result over Ω , and using the boundary conditions (1.3) and (1.4) and definition (2.1) of the space V , we deduce that

$$\begin{aligned} \lambda \iint_{\Omega} \dot{u}_x(x, y, t)(\psi_x(x, y) - \dot{u}_x(x, y, t)) \, dx dy \\ + \mu \iint_{\Omega} \dot{u}_y(x, y, t)(\psi_y(x, y) - \dot{u}_y(x, y, t)) \, dx dy \\ + E \iint_{\Omega} u_x(x, y, t)(\psi_x(x, y) - \dot{u}_x(x, y, t)) \, dx dy \end{aligned} \quad (2.15)$$

$$\begin{aligned}
& +G \iint_{\Omega} u_y(x, y, t)(\psi_y(x, y) - \dot{u}_y(x, y, t)) \, dx dy \\
& = \int_0^L (Gu_y(x, h, t) + \mu \dot{u}_y(x, h, t))(\psi(x, h) - \dot{u}(x, h, t)) \, dx \\
& \quad - \int_0^L (Gu_y(x, -h, t) + \mu \dot{u}_y(x, -h, t))(\psi(x, -h) - \dot{u}(x, -h, t)) \, dx \\
& \quad + \iint_{\Omega} q_B(t)(\psi(x, t) - \dot{u}(x, y, t)) \, dx dy.
\end{aligned}$$

Assume now that $x \in [0, L]$ is fixed. We integrate Eq. (1.2) with respect to y on $[-h, h]$ and deduce that

$$\begin{aligned}
& 2h\mu \dot{w}_{xx}(x, t) + 2hGw_{xx}(x, t) \\
& \quad + (\lambda - \mu) \int_{-h}^h \dot{u}_{xy}(x, y, t) \, dy + (E - G) \int_{-h}^h u_{xy}(x, y, t) \, dy \\
& \quad + \int_{-h}^h f_B(t) \, dy = 0. \tag{2.16}
\end{aligned}$$

Then, using the boundary conditions (1.7), (1.8), and notation

$$\sigma(x, -h, t) = (\lambda - 2\mu)\dot{u}_x(x, -h, t) + (E - 2G)u_x(x, -h, t), \tag{2.17}$$

after some elementary calculus, we find that

$$\begin{aligned}
& (\lambda - \mu) \int_{-h}^h \dot{u}_{xy}(x, y, t) \, dy + (E - G) \int_{-h}^h u_{xy}(x, y, t) \, dy \\
& = f_N(x, t) - \sigma(x, -h, t) + \mu(\dot{u}_x(x, h, t) - \dot{u}_x(x, -h, t)) \\
& \quad + G(u_x(x, h, t) - u_x(x, -h, t)). \tag{2.18}
\end{aligned}$$

Next, we subtract equalities (2.18) and (2.16) to deduce that

$$\begin{aligned}
& -2hGw_{xx}(x, t) - 2h\mu \dot{w}_{xx}(x, t) \\
& = f_N(x, t) - \sigma(x, -h, t) + \mu(\dot{u}_x(x, h, t) - \dot{u}_x(x, -h, t)) \\
& \quad + G(u_x(x, h, t) - u_x(x, -h, t)) + \int_{-h}^h f_B(t) \, dy. \tag{2.19}
\end{aligned}$$

To proceed, we multiply equality (2.19) with $\varphi - \dot{w}$, and then, we integrate the result on $[0, L]$ and perform integration by parts to obtain that

$$\begin{aligned}
& G \iint_{\Omega} w_x(x, t)(\varphi_x(x, t) - \dot{w}_x(x, t)) \, dx dy \\
& \quad + \mu \iint_{\Omega} \dot{w}_x(x, t)(\varphi_x(x, t) - \dot{w}_x(x, t)) \, dx dy, \\
& = \int_0^L -\sigma(x, -h, t)(\varphi(x, t) - \dot{w}(x, t)) \, dx \\
& \quad + \mu \int_0^L (\dot{u}_x(x, h, t) - \dot{u}_x(x, -h, t))(\varphi(x, t) - \dot{w}(x, t)) \, dx \\
& \quad + G \int_0^L u_x(x, h, t) - u_x(x, -h, t))(\varphi(x, t) - \dot{w}(x, t)) \, dx \tag{2.20}
\end{aligned}$$

$$\begin{aligned}
& + 2hG w_x(L, t)(\varphi(L, t) - \dot{w}(L, t)) + 2h\mu \dot{w}_x(L, t)(\varphi(L, t) - \dot{w}(L, t)) \\
& + \int_0^L f_N(\varphi(x, t) - \dot{w}(x, t)) dx + \iint_{\Omega} f_B(\varphi(x, t) - \dot{w}(x, t)) dx dy.
\end{aligned}$$

We now add equalities (2.15) and (2.20) and use integration by parts and the boundary conditions (1.6) and (1.9) to obtain

$$\begin{aligned}
& E \iint_{\Omega} u_x(x, y, t)(\psi_x(x, y) - \dot{u}_x(x, y, t)) dx dy \\
& + G \iint_{\Omega} u_y(x, y, t)(\psi_y(x, y) - \dot{u}_y(x, y, t)) dx dy \\
& + \lambda \iint_{\Omega} \dot{u}_x(x, y, t)(\psi_x(x, y) - \dot{u}_x(x, y, t)) dx dy \\
& + \mu \iint_{\Omega} \dot{u}_y(x, y, t)(\psi_y(x, y) - \dot{u}_y(x, y, t)) dx dy \\
& + G \iint_{\Omega} w_x(x, t)(\varphi_x(x) - \dot{w}_x(x, t)) dx dy \\
& + \mu \iint_{\Omega} \dot{w}_x(x, t)(\varphi_x(x, t) - \dot{w}_x(x, t)) dx dy \\
& + G \iint_{\Omega} w_y(x, t)(\varphi_x(x) - \dot{w}_x(x, t)) dx dy \\
& + \mu \iint_{\Omega} \dot{w}_y(x, t)(\varphi_x(x, t) - \dot{w}_x(x, t)) dx dy, \\
& = \iint_{\Omega} q_B(t)(\psi(x, t) - \dot{u}(x, y, t)) dx dy + \int_0^L f_N(\varphi(x, t) - \dot{w}(x, t)) dx \\
& + \iint_{\Omega} f_B(\varphi(x, t) - \dot{w}(x, t)) dx dy - \int_0^L \sigma(x, -h, t)(\varphi(x, t) - \dot{w}(x, t)) dx \\
& + \int_0^L q_N(x, t)(\psi(x, t) - \dot{u}(x, h, t)) dx - \mu \int_0^L \dot{w}_x(x, t)(\psi(x, t) - \dot{u}(x, h, t)) dx \\
& - G \int_0^L w_x(x, t)(\psi(x, t) - \dot{u}(x, h, t)) dx \\
& + \mu \int_0^L \dot{w}_x(x, t)(\psi(x, t) - \dot{u}(x, -h, t)) dx \\
& + G \int_0^L w_x(x, t)(\psi(x, t) - \dot{u}(x, -h, t)) dx. \tag{2.21}
\end{aligned}$$

In addition, note that

$$\begin{aligned}
& -\mu \int_0^L \dot{w}_x(x, t)(\psi(x, t) - \dot{u}(x, h, t)) dx \tag{2.22} \\
& + \mu \int_0^L \dot{w}_x(x, t)(\psi(x, t) - \dot{u}(x, -h, t)) dx \\
& = -\mu \iint_{\Omega} \dot{w}_x(x, t)(\psi_y(x, t) - \dot{u}_y(x, y, t)) dx dy
\end{aligned}$$

and

$$\begin{aligned}
& -G \int_0^L w_x(x, t)(\psi(x, t) - \dot{u}(x, h, t)) dx \\
& + G \int_0^L w_x(x, t)(\psi(x, t) - \dot{u}(x, -h, t)) dx \\
& = -G \iint_{\Omega} w_x(x, t)(\psi_y(x, t) - \dot{u}_y(x, y, t)) dx dy.
\end{aligned} \tag{2.23}$$

Substituting (2.22) and (2.23) in (2.21) and using the definitions (2.10), (2.11), and (2.13), we obtain

$$\begin{aligned}
& \langle A\dot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X + \langle B\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X \\
& + \int_0^L \sigma(x, -h, t)(\varphi(x, t) - \dot{w}(x, t)) dx \\
& = \langle \mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X \quad \text{for all } \mathbf{v} \in X, \quad t \in [0, T].
\end{aligned} \tag{2.24}$$

Finally, using the boundary condition (1.8) and notation (2.17), it is easy to check that

$$\sigma(x, -h, t)(\varphi(x, t) - \dot{w}(x, t)) dx \leq g|\varphi(x, t)| - g|\dot{w}(x, t)| \quad \forall x \in [0, L].$$

We integrate this inequality on $[0, L]$ and use notation (2.12) to deduce that

$$\int_0^L \sigma(x, -h, t)(\varphi(x, t) - \dot{w}(x, t)) dx \leq j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)). \tag{2.25}$$

We now combine equality (2.24) with inequality (2.25) and then use the initial condition (1.10) and notation (2.14). As a result, we obtain the variational formulation of problem \mathcal{P} .

Problem. \mathcal{P}_V . Find a function $\mathbf{u} : [0, T] \rightarrow X$, such that

$$\begin{aligned}
& \langle A\dot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X + \langle B\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \\
& \geq \langle \mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X \quad \text{for all } \mathbf{v} \in X, \quad t \in [0, T],
\end{aligned} \tag{2.26}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{2.27}$$

Note that Problem \mathcal{P}_V represents an evolutionary variational inequality. Its unique solvability will be presented in the next section. Here, we restrict ourselves to mention that the solution of this inequality will be called a *weak solution* to Problem \mathcal{P} . We also mention that in Sect. 5, we provide a second variational formulation of Problem \mathcal{P} , the so-called dual variational formulation, which, in fact, is equivalent with Problem \mathcal{P}_V .

3. Existence and Uniqueness

Our existence and uniqueness result in the study of Problem \mathcal{P}_V is the following.

Theorem 3.1. Assume (2.5)–(2.9). Then, Problem \mathcal{P}_V has a unique solution with regularity $\mathbf{u} \in C^1([0, T]; X)$.

The proof is carried out in several steps. The first one consists to investigate the properties of the operators A and B and, with this concern, we have the following results.

Lemma 3.2. *Assume that (2.5) holds. Then, the operator A is linear, symmetric continuous, and coercive, i.e., it satisfies*

$$\langle A\mathbf{v}, \mathbf{v} \rangle_X \geq m_A \|\mathbf{v}\|_X^2 \quad \text{for all } \mathbf{v} \in X, \quad \text{with } m_A > 0. \quad (3.1)$$

Lemma 3.3. *Assume that (2.5) holds. Then, the operator B is linear, symmetric, and coercive, i.e., it satisfies*

$$\langle B\mathbf{v}, \mathbf{v} \rangle_X \geq m_B \|\mathbf{u}\|_X^2 \quad \text{for all } \mathbf{v} \in X, \quad \text{with } m_B > 0. \quad (3.2)$$

The proof of Lemmas 3.2 and 3.3 are identical and are based on the standard arguments. Nevertheless, for the convenience of the reader, we present, for instance, the proof of Lemma 3.2.

Proof. The linearity and symmetry of the operator A are obvious. Moreover, an elementary computation shows that

$$\langle A\mathbf{v}, \mathbf{v} \rangle_X \leq (\lambda + 2\mu) \|\mathbf{u}\|_X \|\mathbf{v}\|_X \quad \forall \mathbf{u}, \mathbf{v} \in X, \quad (3.3)$$

which implies that A is continuous. Inequality (3.1) is a direct consequence of the two-dimensional version of Korn's inequality. Indeed, consider an arbitrary element $\mathbf{v} = (\psi(x, y), \varphi(x)) \in X$. Then, the small strain tensor associated with the two-dimensional displacement field \mathbf{v} is given by

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \begin{pmatrix} \psi_x & \frac{1}{2}(\psi_y + \varphi_x) \\ \frac{1}{2}(\psi_y + \varphi_x) & 0 \end{pmatrix}.$$

We have

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2 = \boldsymbol{\varepsilon}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) = \psi_x^2 + \frac{1}{2}(\psi_y + \varphi_x)^2 \quad \text{a.e. on } \Omega. \quad (3.4)$$

Note also that the function \mathbf{v} vanishes on the boundary $x = 0$ of the rectangle Ω which is, obviously, of positive one-dimensional measure and, in addition, since X can be identified as a subspace of $H^1(\Omega)^2$, we have $\mathbf{v} \in H^1(\Omega)^2$. Therefore, using Korn's inequality, we obtain that there exists a constant $c_K > 0$ which depends on Ω , such that

$$\iint_{\Omega} \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2 dx dy \geq c_K \|\mathbf{v}\|_{H^1(\Omega)^2}^2. \quad (3.5)$$

We now combine (3.4) and (3.5) to deduce that

$$\iint_{\Omega} \left(\psi_x^2 + \frac{1}{2}(\psi_y + \varphi_x)^2 \right) dx dy \geq c_K \iint_{\Omega} (\psi^2 + \psi_x^2 + \psi_y^2 + \varphi^2 + \varphi_x^2) dx dy,$$

and then, using (2.2)–(2.4), we obtain that

$$\iint_{\Omega} \left(\psi_x^2 + \frac{1}{2}(\psi_y + \varphi_x)^2 \right) dx dy \geq \tilde{c}_K \|\mathbf{v}\|_X^2, \quad (3.6)$$

where \tilde{c}_K depends on c_K and h . On the other hand, using definition (2.10) of the operator A and inequality (3.6), we deduce that

$$\langle A\mathbf{v}, \mathbf{v} \rangle_X \geq \min(\lambda, 2\mu) \iint_{\Omega} \left(\psi_x^2 + \frac{1}{2}(\psi_y + \varphi_x)^2 \right) dx dy. \quad (3.7)$$

We now combine (3.6), (3.7), and assumption (2.5) to see that inequality (3.1) holds with $m_A = \tilde{c}_K \min(\lambda, 2\mu) > 0$, which concludes the proof. \square

Next, we recall the following existence and uniqueness result for evolutionary variational inequalities.

Lemma 3.4. *Let X be a real Hilbert space, $A : X \rightarrow X$ a strongly monotone Lipschitz continuous operator, $B : X \rightarrow X$ a Lipschitz continuous operator, and $j : X \rightarrow \mathbb{R}$ a convex lower semicontinuous function. Then, given $\mathbf{f} \in C([0, T]; X)$ and $\mathbf{u}_0 \in X$, there exists a unique function $\mathbf{u} \in C^1([0, T]; X)$, such that $\mathbf{u}(0) = \mathbf{u}_0$ and*

$$\begin{aligned} \langle A\dot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X + \langle B\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \\ \geq \langle \mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X \quad \text{for all } \mathbf{v} \in X, \quad t \in [0, T]. \end{aligned} \quad (3.8)$$

Lemma 3.4 is a particular case of Theorem 11.3 in [9], Corollary 7 in [22], and Theorems in [7, 8]. Moreover, it corresponds to Corollary 3.12 in [21]. Nevertheless, for the convenience of the reader, we provide below a sketch of the proof, and for further details, we send the reader to the above-mentioned works.

Proof. The proof is established in several steps as follows.

- (i) In the first step, let $\boldsymbol{\eta} \in C([0, T]; X)$ be given and consider the intermediate problem of finding a function $\mathbf{w}_\eta : [0, T] \rightarrow X$, such that for all $t \in [0, T]$, the inequality below holds:

$$\begin{aligned} \langle A\mathbf{w}_\eta(t), \mathbf{v} - \mathbf{w}_\eta(t) \rangle_X + \langle \boldsymbol{\eta}(t), \mathbf{v} - \mathbf{w}_\eta(t) \rangle_X \\ + j(\mathbf{v}) - j(\mathbf{w}_\eta(t)) \geq \langle \mathbf{f}(t), \mathbf{v} - \mathbf{w}_\eta(t) \rangle_X \quad \forall \mathbf{v} \in X. \end{aligned} \quad (3.9)$$

We prove that this intermediate problem has a unique solution $\mathbf{w}_\eta \in C([0, T]; X)$.

Indeed, using Proposition 31 in [2], it follows from that there exists a unique element $\mathbf{w}_\eta(t)$ that solves (3.9), for each $t \in [0, T]$. To show that $\mathbf{w}_\eta : [0, T] \rightarrow X$ is continuous, consider $t_1, t_2 \in [0, T]$, and for the sake of simplicity in writing, denote $\mathbf{w}_\eta(t_i) = \mathbf{w}_i$, $\boldsymbol{\eta}(t_i) = \boldsymbol{\eta}_i$, and $\mathbf{f}(t_i) = \mathbf{f}_i$ for $i = 1, 2$. We write (3.9) with $t = t_1$ and $\mathbf{v} = \mathbf{w}_2$, then with $t = t_2$ and $\mathbf{v} = \mathbf{w}_1$. Adding the resulting inequalities and using the properties of the operators A and B , we see that

$$\|\mathbf{w}_1 - \mathbf{w}_2\|_X \leq c (\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_X + \|\mathbf{f}_1 - \mathbf{f}_2\|_X). \quad (3.10)$$

Here and below c represent a positive constant, whose value could change from line to line. We deduce from (3.10) that $t \mapsto \mathbf{w}_\eta(t) : [0, T] \rightarrow X$ is a continuous function which proves the existence of the solution. The uniqueness follows from the unique solvability of inequality (3.9) at each $t \in [0, T]$.

- (ii) In the second step, we consider the operator $\Lambda : C([0, T]; X) \rightarrow C([0, T]; X)$ defined by equality

$$\Lambda\boldsymbol{\eta} = B \int_0^t \mathbf{w}_\eta(s) ds + \mathbf{u}_0 \quad \forall \boldsymbol{\eta} \in C([0, T]; X), \quad t \in [0, T], \quad (3.11)$$

and prove that it has a unique fixed point. Indeed, let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C([0, T]; X)$, and denote by \mathbf{w}_i the solution of the variational inequality (3.9) for $\boldsymbol{\eta} = \boldsymbol{\eta}_i$, i.e., $\mathbf{w}_i = \mathbf{w}_{\boldsymbol{\eta}_i}$, $i = 1, 2$. Let $t \in [0, T]$. Then, an argument similar to that in the proof of (3.10) shows that

$$\|\mathbf{w}_1(t) - \mathbf{w}_2(t)\|_X \leq c \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_X. \quad (3.12)$$

We now use definition (3.11) and the properties of the operator B to see that

$$\|\Lambda \boldsymbol{\eta}_1(t) - \Lambda \boldsymbol{\eta}_2(t)\|_X \leq c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_X ds.$$

This inequality shows that, for m large enough, a power Λ^* of Λ is a contraction on the Banach space $C([0, T]; X)$ which concludes this step of the proof.

(iii) Denote by $\mathbf{w}^* \in C([0, T]; X)$, the solution of inequality (3.10) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and let \mathbf{u}^* be the function defined by

$$\mathbf{u}^*(t) = \int_0^t \mathbf{w}^*(s) ds + \mathbf{u}_0 \quad \forall t \in [0, T]. \quad (3.13)$$

Then, it is easy to see that $\mathbf{u}^* \in C^1([0, T]; X)$. Moreover, since $\boldsymbol{\eta}^* = B\mathbf{u}^*$ and $\dot{\mathbf{u}}^* = \mathbf{w}^*$, we deduce from (3.9) that \mathbf{u}^* satisfies (3.8). This proves the existence part of Lemma 3.4. The uniqueness part follows from the uniqueness of the fixed point of the operator Λ or, alternatively, using the Gronwall argument. \square

We are now in a position to provide the proof of Theorem 3.1.

Proof. Using assumption (2.8), it is easy to see that the functional j is a continuous seminorm on the space X . Therefore, it follows that j is a convex lower semicontinuous function on X . In addition, assumptions (2.6) and (2.7) and definition (2.13) imply that $\mathbf{f} \in C([0, T]; X)$. Moreover, assumption (2.9) shows that the initial data has the regularity $\mathbf{u}_0 \in V$. Finally, Lemma 3.2 shows that $A : X \rightarrow X$ is a strongly monotone Lipschitz continuous operator, and Lemma 3.3 implies that $B : X \rightarrow X$ is Lipschitz continuous operator. Theorem 3.1 is now a direct consequence of Lemma 3.4. \square

4. A Continuous Dependence Result

In this section, we study the dependence of the solution with respect to the parameters E , G , and g . To this end, we assume that (2.5)–(2.9) hold and we consider some positive constants E_ρ , G_ρ , and g_ρ which represent a perturbation of E , G , and g , respectively. Here, ρ denotes a positive parameter which will converge to zero. We define the operator B_ρ and the function j_ρ by equalities

$$\langle B_\rho \mathbf{u}, \mathbf{v} \rangle_X = E_\rho \iint_\Omega u_x \psi_x dx dy + G_\rho \iint_\Omega (u_y + w_x)(\psi_y + \varphi_x) dx dy, \quad (4.1)$$

$$j_\rho(\mathbf{v}) = g_\rho \int_0^L |\varphi(x)| dx, \quad (4.2)$$

for all $\mathbf{u} = (u, w)$, $\mathbf{v} = (\psi, \varphi) \in X$. Then, we consider the following variational problem.

Problem. \mathcal{P}_V^ρ . Find a function $\mathbf{u}_\rho : [0, T] \rightarrow X$, such that

$$\begin{aligned} \langle A\dot{\mathbf{u}}_\rho(t), \mathbf{v} - \dot{\mathbf{u}}_\rho(t) \rangle_X + \langle (B_\rho \mathbf{u}_\rho)(t), \mathbf{v} - \dot{\mathbf{u}}_\rho(t) \rangle_X \\ + j_\rho(\mathbf{v}) - j_\rho(\dot{\mathbf{u}}_\rho(t)) \geq \langle \mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_\rho(t) \rangle_X \quad \text{for all } \mathbf{v} \in X, \quad t \in [0, T]. \end{aligned} \quad (4.3)$$

$$\mathbf{u}_\rho(0) = \mathbf{u}_0. \quad (4.4)$$

Using Theorem 3.1, it follows that Problem \mathcal{P}_V has a unique solution $\mathbf{u} \in C^1(0, T; X)$, and, in addition, Problem \mathcal{P}_V^ρ has a unique solution $\mathbf{u}_\rho \in C^1([0, T]; X)$. Our main result in this section is the following.

Theorem 4.1. Assume (2.5)–(2.9) and, moreover, assume that

$$E_\rho \rightarrow E, \quad G_\rho \rightarrow G, \quad g_\rho \rightarrow g \quad \text{as } \rho \rightarrow 0. \quad (4.5)$$

Then, the solution \mathbf{u}_ρ of problem \mathcal{P}_V^ρ converges to the solution \mathbf{u} of the problem \mathcal{P}_V that is

$$\mathbf{u}_\rho \longrightarrow \mathbf{u} \quad \text{in } C^1([0, T]; X) \quad \text{as } \rho \rightarrow 0. \quad (4.6)$$

Proof. Let $\rho > 0$ and let $t \in [0, T]$ be given. We use inequalities (2.26) and (4.3) to deduce that

$$\begin{aligned} \langle A\dot{\mathbf{u}}(t), \dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t) \rangle_X + \langle B\mathbf{u}(t), \mathbf{u}_\rho(t) - \dot{\mathbf{u}}(t) \rangle_X \\ + j(\dot{\mathbf{u}}_\rho(t)) - j(\dot{\mathbf{u}}(t)) \geq \langle \mathbf{f}(t), \dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t) \rangle_X, \\ \langle A\dot{\mathbf{u}}_\rho(t), \dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_\rho(t) \rangle_X + \langle B_\rho \mathbf{u}_\rho(t), \dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_\rho(t) \rangle_X \\ + j_\rho(\dot{\mathbf{u}}(t)) - j_\rho(\dot{\mathbf{u}}_\rho(t)) \geq \langle \mathbf{f}(t), \dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_\rho(t) \rangle_X. \end{aligned}$$

We now add these inequalities and use property (3.1) of the operator A to obtain that

$$\begin{aligned} m_A \|\dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\|_X^2 \leq \langle B_\rho \mathbf{u}_\rho(t) - B\mathbf{u}(t), \dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_\rho(t) \rangle_X \\ + j_\rho(\dot{\mathbf{u}}(t)) - j_\rho(\dot{\mathbf{u}}_\rho(t)) + j(\dot{\mathbf{u}}_\rho(t)) - j(\dot{\mathbf{u}}(t)). \end{aligned} \quad (4.7)$$

Next, we use definitions (4.2) and (2.12) to see that

$$\begin{aligned} j_\rho(\dot{\mathbf{u}}(t)) - j_\rho(\dot{\mathbf{u}}_\rho(t)) + j(\dot{\mathbf{u}}_\rho(t)) - j(\dot{\mathbf{u}}(t)) \\ \leq c |g_\rho - g| \|\dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\|_X, \end{aligned} \quad (4.8)$$

where, here and below, c represents a constant which does not depend on ρ and whose value may change from line to line. We now combine inequalities (4.7) and (4.8) to find that

$$m_A \|\dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\|_X \leq \|B_\rho \mathbf{u}_\rho(t) - B\mathbf{u}(t)\|_X + c |g_\rho - g|. \quad (4.9)$$

On the other hand, using definitions (2.11) and (4.1), it is easy to see that

$$\begin{aligned} \|B_\rho \mathbf{u}_\rho(t) - B\mathbf{u}(t)\|_X \leq (E_\rho + G_\rho) \|\mathbf{u}_\rho(t) - \mathbf{u}(t)\|_X \\ + (|E_\rho - E| + |G_\rho - G|) \|\mathbf{u}(t)\|_X. \end{aligned} \quad (4.10)$$

It follows now from assumption (4.5) that $E_\rho + G_\rho \leq c$ and, therefore, inequalities (4.9) and (4.10) imply

$$\begin{aligned} \|\dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\|_X &\leq c \|\mathbf{u}_\rho(t) - \mathbf{u}(t)\|_X \\ &+ (|E_\rho - E| + |G_\rho - G|) \max_{r \in [0, T]} \|\mathbf{u}(r)\|_X + c |g_\rho - g|. \end{aligned} \quad (4.11)$$

Next, we use the initial conditions (2.27) and (4.3) to see that

$$\|\mathbf{u}_\rho(t) - \mathbf{u}(t)\|_X \leq \int_0^t \|\dot{\mathbf{u}}_\rho(s) - \dot{\mathbf{u}}(s)\|_X ds, \quad (4.12)$$

and then, we substitute this inequality in (4.11) and use Gronwall's lemma to obtain that

$$\begin{aligned} \|\dot{\mathbf{u}}_\rho(t) - \dot{\mathbf{u}}(t)\|_X \\ \leq c (|E_\rho - E| + |G_\rho - G|) \max_{r \in [0, T]} \|\mathbf{u}(r)\|_X + |g_\rho - g|. \end{aligned} \quad (4.13)$$

The convergence (4.6) follows now from inequalities (4.12) and (4.13) and assumption (4.5). \square

5. Dual Variational Formulation

In this section, we introduce and study a second variational formulation of Problem \mathcal{P} , the so-called *dual variational formulation*. It is obtained by operating the change of variable $\boldsymbol{\sigma} = A\dot{\mathbf{u}} + B\mathbf{u}$ in Problem \mathcal{P}_V . Dual variational formulations of boundary problems originate in Contact Mechanics, as explained in [9, 18, 19]. The main idea is to introduce a new variational formulation expressed in terms of the stress field, equivalent with the primal variational formulation which, in turn, is expressed in terms of displacement.

Everywhere below we assume that (2.5)–(2.9) hold and we denote by A^{-1} the inverse of the operator A , whose existence is guaranteed by Lemma 3.2. Note also that Lemmas 3.2 and 3.3 imply that the operators A^{-1} and B linear continuous operators, and we shall use this results in various places below. We start with the following result.

Lemma 5.1. *Then, there exists an operator $\mathcal{R} : C([0, T]; X) \longrightarrow C([0, T], X)$, such that, for any functions $\boldsymbol{\sigma} \in C([0, T]; X)$ and $\mathbf{u} \in C^1([0, T]; X)$ with $\mathbf{u}(0) = \mathbf{u}_0$, the following equivalence holds:*

$$\boldsymbol{\sigma}(t) = A\dot{\mathbf{u}}(t) + B\mathbf{u}(t) \quad \forall t \in [0, T] \quad (5.1)$$

if and only if

$$\dot{\mathbf{u}}(t) = A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t) \quad \forall t \in [0, T]. \quad (5.2)$$

Note that in (5.2), we use the short-hand notation $\mathcal{R}\boldsymbol{\sigma}(t)$ instead of $(\mathcal{R}\boldsymbol{\sigma})(t)$. We shall use this notation in many places below when no confusion arises.

Proof. Let $\boldsymbol{\sigma} \in C([0, T]; X)$ and define the operator $\Lambda_\sigma : C([0, T]; X) \longrightarrow C([0, T]; X)$ by equality

$$(\Lambda_\sigma \boldsymbol{\theta})(t) = -A^{-1}B \left(\int_0^t (\boldsymbol{\theta}(s) + A^{-1}\boldsymbol{\sigma}(s))ds + \mathbf{u}_0 \right) \\ \forall \boldsymbol{\theta} \in C([0, T], X), \quad t \in [0, T]. \quad (5.3)$$

We shall prove that Λ_σ has a unique fixed point, denoted $\mathcal{R}\boldsymbol{\sigma}$. To this end, consider two functions $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in C([0, T]; X)$ and let $t \in [0, T]$. Then, using the properties of the operators A and B , it is easy to see that

$$\|(\Lambda_\sigma \boldsymbol{\theta}_1)(t) - (\Lambda_\sigma \boldsymbol{\theta}_2)(t)\|_X \leq c \int_0^t \|\boldsymbol{\theta}_1(s) - \boldsymbol{\theta}_2(s)\|_X ds,$$

where c denotes a positive constant which depends on A and B . This inequality shows that the operator Λ_σ is a history-dependent operator, and using Theorem 3.1 in [21], we deduce that there exists a unique element $\mathcal{R}\boldsymbol{\sigma} \in C([0, T]; X)$, such that

$$\mathcal{R}\boldsymbol{\sigma}(t) = \Lambda_\sigma(\mathcal{R}\boldsymbol{\sigma})(t). \quad (5.4)$$

We now compare equalities (5.3) and (5.4) to deduce that

$$(\mathcal{R}\boldsymbol{\sigma})(t) = -A^{-1}B \left(\int_0^t (\mathcal{R}\boldsymbol{\sigma}(s) + A^{-1}\boldsymbol{\sigma}(s))ds + \mathbf{u}_0 \right).$$

Assume now that (5.1) holds. Then, it is easy to see that

$$\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = -A^{-1}B\mathbf{u}(t),$$

and since $\mathbf{u}(t) = \int_0^t \dot{\mathbf{u}}(s) ds + \mathbf{u}_0$, we deduce that

$$\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = -A^{-1}B \left(\int_0^t \dot{\mathbf{u}}(s) ds + \mathbf{u}_0 \right),$$

which shows that

$$\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = -A^{-1}B \left(\int_0^t (\dot{\mathbf{u}}(s) - A^{-1}\boldsymbol{\sigma}(s) + A^{-1}\boldsymbol{\sigma}(s)) ds + \mathbf{u}_0 \right). \quad (5.5)$$

We now combine (5.3) and (5.5) to see that $\dot{\mathbf{u}} - A^{-1}\boldsymbol{\sigma}$ is a fixed point for the operator Λ_σ . On the other hand, recall that this operator has a unique fixed point, denoted $\mathcal{R}\boldsymbol{\sigma}$. Therefore, $\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = \mathcal{R}\boldsymbol{\sigma}(t)$, which shows that (5.2) holds. Conversely, assume that (5.2) holds. Then, since $\mathcal{R}\boldsymbol{\sigma}$ is the unique fixed point of the operator Λ_σ , we have the equalities

$$\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = \mathcal{R}\boldsymbol{\sigma}(t) = \Lambda_\sigma(\mathcal{R}\boldsymbol{\sigma})(t) = \Lambda_\sigma(\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t)).$$

We use now definition (5.3) to deduce that (5.5) holds. Next, since

$$\mathbf{u}(t) = \int_0^t \dot{\mathbf{u}}(s) ds + \mathbf{u}_0,$$

equality (5.5) implies that

$$\dot{\mathbf{u}}(t) - A^{-1}\boldsymbol{\sigma}(t) = -A^{-1}B\mathbf{u}(t).$$

This shows that equality (5.1) holds which concludes the proof. \square

Next, for each $t \in [0, T]$, we define the set $\Sigma(t) \subset V$ by equality

$$\Sigma(t) = \{ \boldsymbol{\tau} \in V : \langle \boldsymbol{\tau}, \mathbf{v} \rangle_X + j(\mathbf{v}) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle_X \quad \forall \mathbf{v} \in V \}. \quad (5.6)$$

Then, we consider the following variational problem.

Problem. \mathcal{P}_V^D . Find a function $\boldsymbol{\sigma} : [0, T] \rightarrow X$, such that

$$\boldsymbol{\sigma} \in \Sigma(t), \quad \langle A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t) \rangle_X \geq 0 \quad \text{for all } \boldsymbol{\tau} \in \Sigma(t), \quad t \in [0, T]. \quad (5.7)$$

We refer in what follows to Problem \mathcal{P}_V^D as the dual formulation of Problem \mathcal{P}_V . The link between the variational problems \mathcal{P}_V and \mathcal{P}_V^D is given by the following result.

Theorem 5.2. Assume that (2.5)–(2.9) hold and let $\mathbf{u} \in C^1([0, T]; X)$, $\boldsymbol{\sigma} \in C([0, T]; X)$. Consider the following statements:

- (a) \mathbf{u} is solution to problem \mathcal{P}_V .
- (b) $\boldsymbol{\sigma}$ is solution of problem \mathcal{P}_V^D .
- (c) $\boldsymbol{\sigma} = A\dot{\mathbf{u}} + B\mathbf{u}$ and $\mathbf{u}(0) = \mathbf{u}_0$.

Then, if two of the statements above hold, the reminder one holds, too.

Proof. The proof is based on the implications (a) and (c) \implies (b), (a) and (b) \implies (c), (b) and (c) \implies (a) which will be proved in the three steps below.

- (1) (a) and (c) \implies (b). We assume in what follows that \mathbf{u} solution of \mathcal{P}_V , $\boldsymbol{\sigma} = A\dot{\mathbf{u}} + B\mathbf{u}$, $\mathbf{u}(0) = \mathbf{u}_0$ and let $t \in [0, T]$ be given. Then, Lemma 5.1 implies that $\dot{\mathbf{u}}(t) = A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t)$, and substituting this inequality in (2.26), we have

$$\langle \boldsymbol{\sigma}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq \langle \mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X. \quad (5.8)$$

Next, testing in (5.8) with $\mathbf{v} = 2\dot{\mathbf{u}}(t)$ and $\mathbf{v} = \mathbf{0}_X$, we successively obtain

$$\begin{aligned} \langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t) \rangle_X + j(\dot{\mathbf{u}}(t)) &\geq \langle \mathbf{f}(t), \dot{\mathbf{u}}(t) \rangle_X, \\ \langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t) \rangle_X + j(\dot{\mathbf{u}}(t)) &\leq \langle \mathbf{f}(t), \dot{\mathbf{u}}(t) \rangle_X, \end{aligned}$$

which imply that

$$\langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t) \rangle_X + j(\dot{\mathbf{u}}(t)) = \langle \mathbf{f}(t), \dot{\mathbf{u}}(t) \rangle_X. \quad (5.9)$$

We use (5.8), (5.9), and the definition (5.6), to see that

$$\boldsymbol{\sigma}(t) \in \Sigma(t). \quad (5.10)$$

Moreover, we note that (5.6) and (5.9) yield

$$\langle \boldsymbol{\tau} - \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t) \rangle_X = \langle \boldsymbol{\tau}, \dot{\mathbf{u}}(t) \rangle_X + j(\dot{\mathbf{u}}(t)) - \langle \mathbf{f}(t), \dot{\mathbf{u}}(t) \rangle_X \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(t),$$

and using equality $\dot{\mathbf{u}}(t) = A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t)$, we obtain that

$$\langle \boldsymbol{\tau} - \boldsymbol{\sigma}(t), A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t) \rangle_X \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(t). \quad (5.11)$$

We now gather (5.10) and (5.11) to see that $\boldsymbol{\sigma}$ is solution of Problem \mathcal{P}_V^D , i.e., (b) holds.

2) (a) and (b) \implies (c). We assume in what follows that \mathbf{u} is a solution of \mathcal{P}_V and $\boldsymbol{\sigma}$ is solution of \mathcal{P}_V^D . Denote

$$\tilde{\boldsymbol{\sigma}} = A\dot{\mathbf{u}} + B\mathbf{u} \quad (5.12)$$

and let $t \in [0, T]$. Then, using the implications (a) and (c) \implies (b), it follows that $\tilde{\boldsymbol{\sigma}}$ is solution of \mathcal{P}_V^D . Since both $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are solution to Problem \mathcal{P}_V^D , we have

$$\begin{aligned} \langle A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t), \tilde{\boldsymbol{\sigma}}(t) - \boldsymbol{\sigma}(t) \rangle_X &\geq 0, \\ \langle A^{-1}\tilde{\boldsymbol{\sigma}}(t) + \mathcal{R}\tilde{\boldsymbol{\sigma}}(t), \boldsymbol{\sigma}(t) - \tilde{\boldsymbol{\sigma}}(t) \rangle_X &\geq 0, \end{aligned}$$

and adding these inequalities, we obtain that

$$\langle A^{-1}\tilde{\boldsymbol{\sigma}}(t) - A^{-1}\boldsymbol{\sigma}(t), \boldsymbol{\sigma}(t) - \tilde{\boldsymbol{\sigma}}(t) \rangle_X \leq \langle \mathcal{R}\tilde{\boldsymbol{\sigma}}(t) - \mathcal{R}\boldsymbol{\sigma}(t), \boldsymbol{\sigma}(t) - \tilde{\boldsymbol{\sigma}}(t) \rangle_X.$$

This inequality combined with the properties of A^{-1} yields

$$\|\tilde{\boldsymbol{\sigma}}(t) - \boldsymbol{\sigma}(t)\|_X \leq c \|\mathcal{R}\tilde{\boldsymbol{\sigma}}(t) - \mathcal{R}\boldsymbol{\sigma}(t)\|_X, \quad (5.13)$$

where, here and below, c denotes a given positive constant, whose value will change from line to line. On the other hand, by the definition of the operator \mathcal{R} , we have

$$\begin{aligned} \mathcal{R}\tilde{\boldsymbol{\sigma}}(t) &= -A^{-1}B \left(\int_0^t (\mathcal{R}\tilde{\boldsymbol{\sigma}}(s) + A^{-1}\tilde{\boldsymbol{\sigma}}(s)) ds + \mathbf{u}_0 \right), \\ \mathcal{R}\boldsymbol{\sigma}(t) &= -A^{-1}B \left(\int_0^t (\mathcal{R}\boldsymbol{\sigma}(s) + A^{-1}\boldsymbol{\sigma}(s)) ds + \mathbf{u}_0 \right). \end{aligned}$$

Therefore

$$\begin{aligned} &\|\mathcal{R}\tilde{\boldsymbol{\sigma}}(t) - \mathcal{R}\boldsymbol{\sigma}(t)\|_X \\ &\leq c \left(\int_0^t \|\tilde{\boldsymbol{\sigma}}(s) - \boldsymbol{\sigma}(s)\|_X ds + \int_0^t \|\mathcal{R}\tilde{\boldsymbol{\sigma}}(s) - \mathcal{R}\boldsymbol{\sigma}(s)\|_X ds \right) \end{aligned}$$

and applying Gronwall's lemma, yields

$$\|\mathcal{R}\tilde{\boldsymbol{\sigma}}(t) - \mathcal{R}\boldsymbol{\sigma}(t)\|_X \leq c \int_0^t \|\tilde{\boldsymbol{\sigma}}(s) - \boldsymbol{\sigma}(s)\|_X ds. \quad (5.14)$$

We now combine inequalities (5.13) and (5.14), and then, we apply Gronwal's lemma, again, to deduce that

$$\tilde{\boldsymbol{\sigma}}(t) = \boldsymbol{\sigma}(t). \quad (5.15)$$

It follows now from (5.12) and (5.15) that $\boldsymbol{\sigma} = A\dot{\mathbf{u}} + B\mathbf{u}$ and, therefore, (c) holds.

(3) (b) and (c) \implies (a). We assume that $\boldsymbol{\sigma}$ a solution to problem \mathcal{P}_V^D and, in addition, $\boldsymbol{\sigma} = A\dot{\mathbf{u}} + B\mathbf{u}$ and $\mathbf{u}(0) = \mathbf{u}_0$. Let $t \in [0, T]$. Then, Lemma 5.1 implies that $\dot{\mathbf{u}}(t) = A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t)$. We substitute this equality in (5.7) to obtain

$$\langle \boldsymbol{\tau} - \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t) \rangle_X \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(t). \quad (5.16)$$

Let $\mathbf{d}(t) \in X$ be a subgradient of j in the point $\dot{\mathbf{u}}(t)$. Then

$$j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq \langle \mathbf{d}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X \quad \forall \mathbf{v} \in X, \quad (5.17)$$

and taking successively $\mathbf{v} = \dot{\mathbf{u}}(t)$ and $\mathbf{v} = \mathbf{0}_V$ in this inequality, we find that

$$\langle \mathbf{d}(t), \dot{\mathbf{u}}(t) \rangle_X = j(\dot{\mathbf{u}}(t)). \quad (5.18)$$

We now combine (5.17) and (5.18) to see that

$$j(\mathbf{v}) \geq \langle \mathbf{d}(t), \mathbf{v} \rangle_X \quad \forall \mathbf{v} \in X. \quad (5.19)$$

This inequality shows that $\mathbf{f}(t) - \mathbf{d}(t) \in \Sigma(t)$ and, therefore, we are allowed to test in (5.16) with $\boldsymbol{\tau} = \mathbf{f}(t) - \mathbf{d}(t)$. As a result, we find that

$$\langle \mathbf{f}(t), \dot{\mathbf{u}}(t) \rangle_X \geq \langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t) \rangle_X + \langle \mathbf{d}(t), \dot{\mathbf{u}}(t) \rangle_X, \quad (5.20)$$

and using (5.18) yields

$$\langle \mathbf{f}(t), \dot{\mathbf{u}}(t) \rangle_X \geq \langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t) \rangle_X + j(\dot{\mathbf{u}}(t)). \quad (5.21)$$

Note that the converse inequality also holds, since $\boldsymbol{\sigma}(t) \in \Sigma(t)$. Therefore, we conclude from above that

$$\langle \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t) \rangle_X + j(\dot{\mathbf{u}}(t)) = \langle \mathbf{f}(t), \dot{\mathbf{u}}(t) \rangle_X. \quad (5.22)$$

Now, since $\boldsymbol{\sigma}(t) \in \Sigma(t)$, we have

$$\langle \boldsymbol{\sigma}(t), \mathbf{v} \rangle_X + j(\mathbf{v}) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle_X \quad \forall \mathbf{v} \in X,$$

and using (5.21), we deduce that

$$\langle \boldsymbol{\sigma}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq \langle \mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_X \quad \forall \mathbf{v} \in X.$$

Finally, using equalities $\boldsymbol{\sigma}(t) = A\dot{\mathbf{u}}(t) + B\mathbf{u}(t)$ and $\mathbf{u}(0) = \mathbf{u}_0$, we deduce that \mathbf{u} is a solution of Problem \mathcal{P}_V , which concludes the proof. \square

A careful examination of Problems \mathcal{P}_V and \mathcal{P}_V^D leads to the conclusion that these problems have a different feature. First, Problem \mathcal{P}_V is an evolutionary variational inequality, since the derivative of the unknown \mathbf{u} appears in its statement. Therefore, an initial condition, (2.27), is required. Moreover, it does not involve any constraint on the solution. In contrast, Problem \mathcal{P}_V^D is a history-dependent inequality with constraints. Indeed, this inequality is governed by the operator \mathcal{R} which satisfies inequality (5.14) and, therefore, is a history-dependent operator. Moreover, the inequality is governed by the set of constraints $\Sigma(t)$, which is a time-dependent convex set. Nevertheless, despite these different features, Problems \mathcal{P}_V and \mathcal{P}_V^D are equivalent, as stated in Theorem 5.2. Moreover, combining Theorem 5.2 with Theorem 3.1, we deduce the unique solvability of Problem \mathcal{P}_V^D , under assumptions (2.5)–(2.9).

References

- [1] Baiocchi, C., Capelo, A.: Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems. Wiley, Chichester (1984)
- [2] Brézis, H.: Equations et inéquations non linéaires dans les espaces vectoriels en dualité. Ann. Inst. Fourier **18**, 115–175 (1968)
- [3] Duvaut, G., Lions, J.-L.: Inequalities in Mechanics and Physics. Springer, Berlin (1976)

- [4] Glowinski, R.: Numerical Methods for Nonlinear Variational Problems. Springer, New York (1984)
- [5] Glowinski, R., Lions, J.-L., Trémolières, R.: Numerical Analysis of Variational Inequalities. North-Holland, Amsterdam (1981)
- [6] Han, W., Reddy, B.D.: Plasticity: Mathematical Theory and Numerical Analysis. Springer, New York (1999)
- [7] Han, W., Sofonea, M.: Evolutionary variational inequalities arising in viscoelastic contact problems. *SIAM J. Numer. Anal.* **38**, 556–579 (2000)
- [8] Han, W., Sofonea, M.: Time-dependent variational inequalities for viscoelastic contact problems. *J. Comput. Appl. Math.* **136**, 369–387 (2001)
- [9] Han, W., Sofonea, M.: Quasistatic contact problems in viscoelasticity and viscoplasticity. In: *Studies in Advanced Mathematics*, vol. 30. American Mathematical Society, Providence, International Press, Somerville (2002)
- [10] Han, W., Migórski, S., Sofonea, M. (eds.) Advances in variational and hemivariational inequalities. In: *Advances in Mechanics and Mathematics*, vol. 33. Springer, New York (2015)
- [11] Hlaváček, I., Haslinger, J., Necăs, J., Lovíšek, J.: Solution of Variational Inequalities in Mechanics. Springer, New York (1988)
- [12] Kikuchi, N., Oden, J.T.: Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods. SIAM, Philadelphia (1988)
- [13] Kinderlehrer, D., Stampacchia, G.: An introduction to variational inequalities and their applications. In: *Classics in Applied Mathematics*, vol. 31. SIAM, Philadelphia (2000)
- [14] Lions, J.-L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Gauthiers-Villars, Paris (1969)
- [15] Migórski, S., Shillor, M., Sofonea, M.: Nonlinear analysis series B: real world applications. *Contact Mech. (Special Issue)* **22**, 435–679 (2015)
- [16] Nečas, J., Hlaváček, I.: Mathematical Theory of Elastic and Elastico-Plastic Bodies: An Introduction. Elsevier, Amsterdam (1981)
- [17] Panagiotopoulos, P.D.: Inequality Problems in Mechanics and Applications. Birkhäuser, Boston (1985)
- [18] Shillor, M., Sofonea, M., Telega, J.J.: Models and analysis of quasistatic contact. In: *Lecture Notes in Physics*, vol. 655. Springer, Berlin (2004)
- [19] Sofonea, M., Matei, A.: Variational inequalities with applications. A study of antiplane frictional contact problems. In: *Advances in Mechanics and Mathematics*, vol. 18. Springer, New York (2009)
- [20] Sofonea, M., Matei, A.: History-dependent quasivariational inequalities arising in contact mechanics. *Eur. J. Appl. Math.* **22**, 471–491 (2011)
- [21] Sofonea, M., Matei, A.: Mathematical models in contact mechanics. In: *London Mathematical Society Lecture Note Series*, vol. 398. Cambridge University Press, Cambridge (2012)
- [22] Sofonea, M., Xiao, Y.: Fully history-dependent quasivariational inequalities in contact mechanics. *Appl. Anal.* doi:[10.1080/00036811.2015.1093623](https://doi.org/10.1080/00036811.2015.1093623) (published online, 7 October 2015)

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