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Principal components analysis and cyclostationarity

Alain BOUDOU[a], Sylvie VIGUIER-PLA[b,a,*]

[a]*Equipe de Stat. et Proba., Institut de Mathématiques, UMR5219, Université Paul Sabatier, 118 Route de Narbonne, F-31062 Toulouse Cedex 9, France*
[b]*Université de Perpignan via Domitia, LAMPS, 52 av. Paul Alduy, 66860 Perpignan Cedex 9, France*

Abstract

In this paper, we give a definition of a cyclostationary function, which specifies and extends usual definitions of cyclostationary processes. We transform such a cyclostationary function into a series. The property of stationarity of the series lets us proceed to the Principal Components Analysis in the frequency domain. This technique requires the introduction of new notions as the conjugated of a spectral measure, the association of a set of unitary operators with a family of stationary series, and the ampliation. We illustrate this work by a simulated example, and we end by a particular case of cyclostationary function, where the Principal Components Analysis in the frequency domain is equivalent to the classical Principal Components Analysis.

Keywords: Cyclostationarity, Orthogonal projectors, Principal Components Analysis, Random measures, Spectral measures, Stationary processes, Unitary operators.
2020 MSC: Primary 60G57, Secondary 60G10, 60B15, 60H05

1. Introduction

Cyclostationary processes, also refered as periodically correlated random processes, have been explored since the 1960s and before (Voychishin and Dragan [17], is an English translation of articles first published in 1957 and 1960). It has been firstly mathematically treated by Gladyshev [9], and largely developed by Hurd [10], as a modelling of phenomena which are periodically correlated, that is as processes for which some statistics present a periodicity. A renewed interest of mathematical aspects of this field can be observed, as in Bouleux et al. [7], where we find a characterization of these processes using dilation matrices. Periodicity occurs in various phenomena, due for example to modulation in signal theory, rotation in mechanics, revolution of planets or pulsation of stars for astronomy, seasonality in economics, or sanguine pulse for medicine. This notion has already been largely employed in applications. Let us cite some examples such as telecommunications (Gardner [8]), mechanic transmission (Randall et al. [12]), radioastronomy (Weber and Faye [16]), locomotion (Zakaria [18]), or medical studies (Roussel [13]). A collection of illustrations can be found in Antoni [1].

Many authors address cyclostationary signals on a temporal level, indexed by $\mathbb{R}$ or $\mathbb{Z}$. In that case, the shape of the process is easy to visualize and to model. In the present paper, we give a definition of cyclostationarity for random functions (r.f.'s) indexed by $\mathbb{R}^k$, and we propose a way for processing the Principal Components Analysis (PCA) of such r.f.'s. Indeed, we can imagine phenomena varying on both space and time, as, for example, a flow in fluid mechanics. This kind of cyclostationarity gives tools for modelling such phenomena, and many other types.

Very often, when considering a process, it is centered, so the scalar product becomes a covariance. We do not make this hypothesis, because it is not necessary for the mathematical development, and this simplifies the writing. Nevertheless, this hypothesis can be done, if it makes more sense from a statistical point of view.

Section 2 is devoted to mathematical recalls. We work in the complex field, to be able to use Fourier transform, as we address signals in the frequency domain.

*Corresponding author. Email address: viguier@univ-perp.frhttps://www.math.univ-toulouse.fr/~viguier/

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The stochastic integral is defined as an isometry. The spectral measure is exposed as a mapping defined on a set which is a $\sigma-$field, what is natural as it is a measure, taking values in a set of projectors, that is idempotent mappings. We then introduce the association between a unitary operator and a spectral measure. Let us expose hereafter a particular case for which the notions are easy to write. We consider a $\mathbb{C}-$Hilbert space H, which can be a space of random variables, where mappings operate. If $\{P_j; j \in J\}$ is a finite family of projectors from H into H such that $\sum_{j\in J} P_j = I$, then $U = \sum_{j\in J} \mathrm{e}^{i\lambda_j} P_j$ is a unitary operator for which the associated spectral measure is $\mathcal{E} = \sum_{j\in J} \delta_{\lambda_j}(.)P_j$, where δ_{λ_j} is the Dirac measure concentrated on λ_j, element from $[-\pi;\pi[$. If X is an element from H, we note that $(U^n(X))_{n\in\mathbb{Z}}$ (we further denote $U^n(X) = U^n X$) is a stationary series, because $\langle U^n X, U^m X\rangle = \langle U^{n+k}X, U^{m+k}X\rangle$, for which the associated random measure is $Z^X_{\mathcal{E}}(.) = \sum_{j\in J} \delta_{\lambda_j}(.)P_j X$, what means that $U^n X = \int \mathrm{e}^{i.n}\mathrm{d}Z^X_{\mathcal{E}}(.) = \sum_{j\in J} \mathrm{e}^{i\lambda_j n} P_j X$.

We end the section by the recall of the PCA in the frequency domain. We say that a $k-$dimensional series $(X_n)_{n\in\mathbb{Z}}$ is $\mathbb{C}^k-$stationary when $\mathrm{E}(X_n {}^t\overline{X}_m) = \mathrm{E}(X_{n-m} {}^t\overline{X}_0)$ (or more generally, $\mathrm{E}(X_m \otimes X_n) = \mathrm{E}(X_0 \otimes X_{n-m})$, where $\otimes$ stands for the tensor product), for any pair (n, m) of elements from $\mathbb{Z}$. The first p steps ($p < k$) of the PCA in the frequency domain of this series may be presented as the search of a $p-$dimensional filter $(\sum_m C_m X_{n-m})_{n\in\mathbb{Z}}$ (C_m being an operator from $\mathbb{C}^k$ into $\mathbb{C}^p$) which summarizes $(X_n)_{n\in\mathbb{Z}}$. In order to measure the quality of the resulting summary, we transform the $p-$dimensional series $(\sum_m C_m X_{n-m})_{n\in\mathbb{Z}}$ into a $k-$dimensional series thanks to a second filtering operation, $(\sum_q R_q \sum_m C_m X_{n-q-m})_{n\in\mathbb{Z}} = (\sum_l (\sum_q R_q C_{l-q}) X_{n-l})_{n\in\mathbb{Z}}$ (R_q is an operator from $\mathbb{C}^p$ into $\mathbb{C}^k$, and then $\sum_q R_q C_{l-q}$ is an operator from $\mathbb{C}^k$ into $\mathbb{C}^k$). This last series is a reconstitution of the data. It is stationarily correlated with $(X_n)_{n\in\mathbb{Z}}$: $\mathrm{E}(X_m {}^t\overline{\sum_l (\sum_q R_q C_{l-q}) X_{n-l}}) = \mathrm{E}(X_{m-n} {}^t\overline{\sum_l (\sum_q R_q C_{l-q}) X_{-l}})$, and then $\|X_n - \sum_l (\sum_q R_q C_{l-q}) X_{n-l}\| = \|X_0 - \sum_l (\sum_q R_q C_{l-q}) X_{-l}\|$. The filters are chosen such that this last quantity is as small as possible.

In Sections 3 to 6, we develop mathematical tools which we are going to use in the studies of cyclostationary functions. We define the conjugate spectral measure. We associate a spectral measure with a family of stationary series. We study the ampliation, operation which consists, from an operator of H (mapping from H into H), to define an operator of $L^2_H(\Lambda)$ (space of square-integrable mappings from Λ into H). All these mathematical tools will be necessary for Section 7, where we study the cyclostationarity. Section 8 is devoted to a simulated example. We end by an exploration of a particular case of cyclostationary function, which is decomposed into the product of two independant random variables, one of them is stationary, and the other one is periodic.

Let us expose shortly the case of cyclostationary series.

The set of the random variables $\{x_n; n \in \mathbb{Z}\}$ is a $\{0, \ldots, k-1\}-$cyclostationary series when $\mathrm{E}(x_n \overline{x_m}) = \mathrm{E}(x_{n+pk}\overline{x_{m+pk}})$, for any (n, m, p) from $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. So we can easily verify that $\{(x_{kn+p})_{n\in\mathbb{Z}}; p \in \{0, \ldots, k-1\}\}$ is a family of stationary series pairwise stationarily correlated. The subset $\{0, \ldots, k-1\}$ is such that with any n from $\mathbb{Z}$ we can associate an element, and only one, from $\{0, \ldots, k-1\} \times k\mathbb{Z}$, $(n - k[\frac{n}{k}], k[\frac{n}{k}])$ such that $n = n - k[\frac{n}{k}] + k[\frac{n}{k}]$ ($[x]$ is the integer part of x).

If we consider the $k-$dimensional random vector $X_n^T = \begin{pmatrix} x_{nk} & \cdots & x_{nk+k-1}\end{pmatrix}$, we can easily verify that $(X_n)_{n\in\mathbb{Z}}$ is a $\mathbb{C}^k-$stationary series. So we can define the PCA in the frequency domain of $(X_n)_{n\in\mathbb{Z}}$. The first p steps give a $\mathbb{C}^p-$stationary series, $(X'_n)_{n\in\mathbb{Z}}$, and a $\mathbb{C}^k-$stationary series, $(X''_n)_{n\in\mathbb{Z}}$. Each of the random vectors X''_n is $k-$dimensional: $(X''_n)^T = \begin{pmatrix} y_{n,1} & \cdots & y_{n,k}\end{pmatrix}$. We can then verify that $\{x''_n; n \in \mathbb{Z}\}$, where $x''_n = y_{[\frac{n}{k}], n-k[\frac{n}{k}]+1}$, is a $\{0, \ldots, k-1\}-$cyclostationary series.

We show that

$$\sum_{l=0}^{k-1} \|x_{nk+l} - x''_{nk+l}\|^2 = \|X_n - X''_n\|^2 = \|X_0 - X''_0\|^2 = \sum_{l=0}^{k-1} \|x_l - x''_l\|^2. \tag{1}$$

So we can summarize a cyclostationary series by a $\mathbb{C}^p-$stationary series, $(X'_n)_{n\in\mathbb{Z}}$. The quantity (1) is a measure of the quality of this summary. As for the cyclostationary series $(X''_n)_{n\in\mathbb{Z}}$, it enables the reconstruction of the data.

Let us now present a particular case of the foregoing study. This case is similar to the previous one, but concerns the functions instead of series. Let us consider a family of integrable square module random variables $\{X_t; t \in \mathbb{R}\}$ such that $\langle X_t, X_{t'}\rangle = \langle X_{t+1}, X_{t'+1}\rangle$, for any pair (t, t') of reals.

From the process $(X_t)_{t\in\mathbb{R}}$, we build a stationary series $(Y_n)_{n\in\mathbb{Z}}$, each element Y_n is a random vector, more precisely, a random variable taking values in the $\mathbb{C}-$Hilbert space $L^2([0; 1[)$ (which substitutes to $\mathbb{C}^k$), which is defined from the family of random variables $\{X_{t+n}; t \in [0; 1[\}$ (this family substitutes to the family $\{x_{nk}, \ldots, x_{(n+1)k-1}\}$ in the previous example).

The series $(Y_n)_{n\in\mathbb{Z}}$ is such that $\mathrm{E}(Y_n \otimes Y_m) = \mathrm{E}(Y_{n-m} \otimes Y_0)$ and we can proceed to the PCA in the frequency domain.

2. Prerequisite

The goal of this section is to define the notation and to recall the mathematical tools for the understanding of this text. Recalls of Subsection 2.1 to Subsection 2.5 come from Boudou [3] and Boudou and Romain [5].

2.1. Measurable spaces and Hilbert spaces

All along this text, k denotes a non null positive integer (hence possibly equal to 1). When H is a $\mathbb{C}-$Hilbert space, we denote by $\mathcal{P}(H)$ the set of the projectors of H. The adjoint of a continuous operator L is denoted L^*. If H and H' are two separable $\mathbb{C}-$Hilbert spaces, we denote by $\sigma_2(H,H')$ the family of the Hilbert-Schmidt operators from H in H', which is also a $\mathbb{C}-$Hilbert space, for which the scalar product is $\langle K,L\rangle_{\sigma^2} = \mathrm{tr}\, KL^*$. For any (K,h,h') of $\sigma_2(H,H')\times H\times H'$, we have $\langle K, h\otimes h'\rangle_{\sigma^2} = \langle Kh, h'\rangle$.

In this text, $(\Omega,\mathcal{A},P)$ and (Λ,ξ,η) are two probability spaces such that the $\mathbb{C}-$Hilbert spaces $L^2(\Omega,\mathcal{A},P)$, shortly denoted, when there is no ambiguity, $L^2(\Omega)$, and $L^2(\Lambda,\xi,\eta)$, shortly denoted $L^2(\Lambda)$, are separable. According to the context, $\overline{x}$ stands for the conjugate complex of x or for the coset of x. Then, an element X from $\mathcal{L}^2_H(\Omega,\mathcal{A},P)$ (resp. of $\mathcal{L}^2_H(\Lambda,\xi,\eta)$) is a measurable mapping from Ω (resp. from Λ) to the $\mathbb{C}-$Hilbert space H, of $P-$integrable (resp. $\eta-$integrable) square norm. The coset $\overline{X}$ is an element from the $\mathbb{C}-$Hilbert space $L^2_H(\Omega,\mathcal{A},P)$, shortly denoted $L^2_H(\Omega)$ (resp. $L^2_H(\Lambda,\xi,\eta)$, shortly denoted $L^2_H(\Lambda)$). We will say that X is a representative of $\overline{X}$. So when $X = X'$ $P-$almost everywhere (resp. $\eta-$almost everywhere), then $\overline{X} = \overline{X'}$. When there is no ambiguity, the symbol $\circ$ of composition between two mappings will be omitted, as well as the parentheses for an element to which a mapping is applied (for example, $K\circ L$ will be denoted KL, and $K(h)$ will be denoted Kh).

The mapping $\gamma : y\in L^2(\Omega)\mapsto \overline{y}\in L^2(\Omega)$ (resp. $\Gamma : h\in L^2(\Lambda)\mapsto \overline{h}\in L^2(\Lambda)$) is involutive, antilinear, and preserves the norm. This implies that $\langle y_1,y_2\rangle = \langle \gamma y_2,\gamma y_1\rangle$ (resp. $\langle h_1,h_2\rangle = \langle \Gamma h_2,\Gamma h_1\rangle$), for any pair (y_1,y_2) of elements from $L^2(\Omega)$ (resp. (h_1,h_2) of elements from $L^2(\Lambda)$).

The $\sigma-$field $\mathcal{B}_{\Pi^k}$ of subsets of Π^k ($\Pi = [-\pi;\pi[$), stands for the trace of $\mathcal{B}_{\mathbb{R}^k}$, Borel $\sigma-$field of $\mathbb{R}^k$, on Π^k.

We denote by W_Π the mapping $\alpha\in\Pi\mapsto -\alpha - 2\pi\left[\frac{-\alpha+\pi}{2\pi}\right]\in\Pi$, by W_{Π^k} the mapping $(\alpha_1,\ldots,\alpha_k)\in\Pi^k\mapsto (W_\Pi\alpha_1,\ldots,W_\Pi\alpha_k)\in\Pi^k$, and by P_j, $j\in\{1,\ldots,k\}$, the coordinate mapping: $(\alpha_1,\ldots,\alpha_k)\in\Pi^k\mapsto\alpha_j\in\Pi$. The mapping P_j is measurable (because for any A from $\mathcal{B}_\Pi$, $P_j^{-1}A = \Pi\times\cdots\times A\times\cdots\times\Pi\in\mathcal{B}_\Pi\otimes\cdots\otimes\mathcal{B}_\Pi = \mathcal{B}_{\Pi^k}$), and so is it for W_Π (because for any A of $\mathcal{B}_\Pi$, we have $W_\Pi^{-1}A = (-A\cap]-\pi;\pi[)\cup(A\cap\{-\pi\})$). If we remark that $P_j\circ W_{\Pi^k} = W_\Pi\circ P_j$, we can deduce the measurability of W_{Π^k}.

2.2. Random measure and stochastic integral

In this subsection, (E,ζ) and $(F,\mathcal{F})$ stand for two measurable spaces, H and H' stand for two $\mathbb{C}-$Hilbert spaces.

We start by the definition of the well-known random measure.

Definition 1. A random measure (r.m.) Z, defined on ζ and taking values on H, is a mapping from ζ in H such that:
(i) for any pair (A,A') of disjoint elements of ζ, $Z(A\cup A') = ZA + ZA'$ and $\langle ZA, ZA'\rangle = 0$;
(ii) for any sequence $(A_n)_{n\in\mathbb{N}}$ which decreasingly converges to $\emptyset$, $\lim ZA_n = 0$.

The "orthogonality" condition of ZA and ZA', as soon as $A\cap A' = \emptyset$, implies the following.

Proposition 1. *The mapping $\mu_Z : A\in\zeta\mapsto\|ZA\|^2\in\mathbb{R}_+$ is a finite measure.*

For any set E, we denote by $\mathrm{vect}E$ the space generated by the elements from E, and by $\overline{\mathrm{vect}E}$ its closure. For any pair (A,A') of elements from ζ, we have $\langle ZA,ZA'\rangle = \langle 1_A, 1_{A'}\rangle_{L^2(\mu_Z)}$, and, as $\overline{\mathrm{vect}}\{1_A; A\in\zeta\} = L^2(E,\zeta,\mu_Z)$, we can define the stochastic integral with respect to the r.m. Z.

Definition 2. The stochastic integral with respect to the r.m. Z is the unique isometry from $L^2(E,\zeta,\mu_Z)$ on $H_Z = \overline{\mathrm{vect}}\{1_A; A\in\zeta\}$, which associates ZA with 1_A, for any A from ζ.

The image of an element φ from $L^2(E,\zeta,\mu_Z)$ by this isometry is named stochastic integral of φ with respect to the r.m. Z and is denoted $\int\varphi\mathrm{d}Z$. The composition of a r.m. with a bijection gives a new stochastic integral as follows:

Proposition 2. *Let $\mathcal{L}$ be a linear (resp. antilinear) bijection from H on H', which preserves the norm. We have*
(i) *$\mathcal{L} \circ Z$ is a r.m.;*
(ii) *$\mu_{\mathcal{L}\circ Z} = \mu_Z$;*
(iii) *for any φ from $L^2(\mu_{\mathcal{L}\circ Z})$, $\int \varphi d\mathcal{L} \circ Z = \mathcal{L}(\int \varphi dZ)$ (resp. $\int \varphi d\mathcal{L} \circ Z = \mathcal{L}(\int \overline{\varphi} dZ)$).*

Just as we can define the image of a probability measure, we can define the image of the r.m. Z.

Proposition 3. *If f is a measurable function from E into F, then*
(i) *the mapping $f(Z) : A \in \mathcal{F} \mapsto Zf^{-1}A \in H$ is a r.m. named image of Z by f;*
(ii) *the image of μ_Z by f equals $\mu_{f(Z)}$;*
(iii) *if φ belongs to $L^2(F, \mathcal{F}, \mu_{f(Z)})$, then $\varphi \circ f$ belongs to $L^2(E, \zeta, \mu_Z)$ and $\int \varphi df(Z) = \int \varphi \circ f dZ$.*

Let us notice that, when $H = L^2(\Omega, \mathcal{A}, P)$, the qualification of random for these measures takes its full meaning, because $Z(A)$, element of $L^2(\Omega)$, is a random variable.

2.3. Stationary series

Let us first define a stationary series taking values in a $\mathbb{C}-$Hilbert space H.

Definition 3. A stationary series taking values in H is a family $(X_n)_{n\in\mathbb{Z}^k}$ of elements from H such that $\langle X_n, X_m\rangle = \langle X_{n-m}, X_0\rangle$, for any pair (n, m) of elements from $\mathbb{Z}^k$.

When $k = 1$ and $H = L^2(\Omega, \mathcal{A}, P)$, if $\mathrm{E}X_n = 0$, we get the usual definition of the wide sense of the stationarity, because $\langle X_n, X_m\rangle = \mathrm{cov}\,(X_n, X_m)$. With a stationary series of elements from H, we can associate a r.m. This r.m. is defined on $\mathcal{B}_{\Pi^k}$. For any $(n_1, \ldots, n_k)$ of $\mathbb{Z}^k$, we denote by $\mathrm{e}^{\langle .,(n_1,\ldots,n_k)\rangle}$ the measurable mapping which, to $(\alpha_1, \ldots, \alpha_k)$ from Π^k associates the complex $\mathrm{e}^{i(\alpha_1 n_1 + \ldots + \alpha_k n_k)}$. Of course, in the particular case where $k = 1$, we have $\mathrm{e}^{\langle .,n\rangle} : \alpha \in \Pi \mapsto \mathrm{e}^{i\alpha n} \in \mathbb{C}$, for any n of $\mathbb{Z}$. Any stationary series is a Fourier transform of a r.m.

Proposition 4. *If $(X_n)_{n\in\mathbb{Z}^k}$ is a stationary series of elements from H, there exists a r.m. Z, and only one, defined on $\mathcal{B}_{\Pi^k}$, and taking values in H, such that $X_n = \int e^{i\langle .,n\rangle} dZ$, for any n of $\mathbb{Z}^k$.*

Then it is natural to consider the r.m. associated with a stationary series.

Definition 4. We name r.m. associated with a stationary series $(X_n)_{n\in\mathbb{Z}^k}$ the unique r.m. such that $X_n = \int \mathrm{e}^{i\langle .,n\rangle} \mathrm{d}Z$, for any n from $\mathbb{Z}^k$.

Proposition 4 has got a converse.

Proposition 5. *If Z is a r.m. defined on $\mathcal{B}_{\Pi^k}$, $(\int e^{i\langle .,n\rangle} dZ)_{n\in\mathbb{Z}^k}$ is a stationary series of associated r.m. Z.*

In statistics, we often use the notion of stationarily correlated processes.

Definition 5. Two stationary series $(X_n)_{n\in\mathbb{Z}^k}$ and $(X'_n)_{n\in\mathbb{Z}^k}$ are stationarily correlated when $\langle X_n, X'_n\rangle = \langle X_{n-n'}, X'_0\rangle$, for any pair (n, n') of elements from $\mathbb{Z}^k$.

The stationary correlation can also express in the frequency domain.

Definition 6. Two r.m.'s Z and Z' defined on $\mathcal{B}_{\Pi^k}$ taking values in H are said to be stationarily correlated when $\langle ZA, Z'A\rangle = 0$, for any pair (A, A') of disjoint elements from $\mathcal{B}_{\Pi^k}$.

We have then the following equivalence.

Proposition 6. *Two stationary series $(X_n)_{n\in\mathbb{Z}^k}$ and $(X'_n)_{n\in\mathbb{Z}^k}$, taking values in H, are stationarily correlated if and only if their respective associated r.m.'s Z and Z' are stationarily correlated.*

2.4. Spectral measure

In this subsection, (E, ζ), $(F, \mathcal{F})$ and H have the same meaning as in Section 2.2. Let us start by the definition of the spectral measure.

Definition 7. A spectral measure (s.m.) on ζ for H is a mapping $\mathcal{E}$, from ζ into $\mathcal{P}(H)$, such that
(i) $\mathcal{E}E = I_H$;
(ii) $\mathcal{E}(A \cup B) = \mathcal{E}A + \mathcal{E}B$, for any pair (A, B) of disjoint elements from ζ;
(iii) $\lim_n \mathcal{E}A_n X = 0$, for any sequence $(A_n)_{n\in\mathbb{N}}$ of elements from ζ which decreasingly converge to $\emptyset$, and for any X from H.

With a s.m., we can define a r.m. for any X from H as follows.

Proposition 7. *If $\mathcal{E}$ is a s.m. on ζ for H, then, for any X from H, the mapping $Z^X_{\mathcal{E}} : A \in \zeta \mapsto \mathcal{E}AX \in H$ is a r.m.*

Conversely, from a family of r.m.'s, we can define a s.m.

Proposition 8. *If $\{Z^X; X \in H\}$ is a family of r.m.'s, defined on ζ and taking values in H, such that*
(a) $Z^X E = X$, for any X from H,
(b) for any pair (X, X') of elements from H, the r.m.'s Z^X and $Z^{X'}$ are stationarily correlated,
we can state that
(i) *for any A from ζ, the mapping $\mathcal{E}A : X \in H \mapsto Z^X A \in H$ is a projector;*
(ii) *the mapping $\mathcal{E} : A \in \zeta \mapsto \mathcal{E}A \in \mathcal{P}(H)$ is a s.m. on ζ for H.*

Let us consider a measurable mapping f from E into F and $\mathcal{E}$ a s.m. on ζ for H. We can define the image of $\mathcal{E}$ by f.

Proposition 9. *The mapping $f\mathcal{E} : A \in \mathcal{F} \mapsto \mathcal{E}f^{-1}A \in \mathcal{P}(H)$ is a s.m. on $\mathcal{F}$ for H, named image of $\mathcal{E}$ by f. Moreover, for any X from H, $Z^X_{f\mathcal{E}} = f(Z^X_{\mathcal{E}})$.*

2.5. Association between spectral measure and unitary operator

A s.m. on $\mathcal{B}_\Pi$ for the $\mathbb{C}-$Hilbert space H can be associated with a unitary operator of H in a bijective way.

Proposition 10. *With any unitary operator U of H, we can associate a s.m. $\mathcal{E}$, and only one, on $\mathcal{B}_\Pi$ for H, such that $UX = \int e^{i.1} dZ^X_{\mathcal{E}}$, for any X from H.*

So we can define the s.m. associated with a unitary operator.

Definition 8. We name s.m. associated with a unitary operator U of H, the unique s.m. $\mathcal{E}$, on $\mathcal{B}_\Pi$ for H, such that $UX = \int \mathrm{e}^{i.1} \mathrm{d}Z^X_{\mathcal{E}}$, for any X from H.

The converse of Proposition 10 can be expressed as follows.

Proposition 11. *If $\mathcal{E}$ is a s.m. on $\mathcal{B}_\Pi$ for H, then the mapping $X \in H \mapsto \int e^{i.1} dZ^X_{\mathcal{E}} \in H$ is a unitary operator of associated s.m. $\mathcal{E}$.*

Thanks to these properties, we extend the field of validity of this association.

Definition 9. Let $\mathcal{E}$ be a s.m. on $\mathcal{B}_{\Pi^k}$ for H, and let us denote by U_j the unitary operator of H of associated s.m. $P_j\mathcal{E}$. We name group of the unitary operators of H deduced from the s.m. $\mathcal{E}$ the family of unitary operators $\{U_n; n \in \mathbb{Z}^k\}$ where, for any $(n_1, \ldots, n_k)$ from $\mathbb{Z}^k$, $U_{(n_1,\ldots,n_k)} = \prod_{j=1}^k U_j^{n_j}$.

So we have the following property.

Proposition 12. *If $\{U_n; n \in \mathbb{Z}^k\}$ is the group of the unitary operators deduced from $\mathcal{E}$, s.m. on $\mathcal{B}_{\Pi^k}$ for H, then*
(i) *$U_0 = I_H$;*
(ii) *for any pair (n, m) from $\mathbb{Z}^k$, $U_n U_m = U_{n+m}$;*
(iii) *$(U_n)^* = U_{-n}$, for any n from $\mathbb{Z}^k$;*
(iv) *$(U_n X)_{n\in\mathbb{Z}^k}$ is a stationary series of associated r.m. $Z^X_{\mathcal{E}}$, this for any X from H.*

Let us finally examine some relations which we will use in Section 6. For any integer $p \geq 1$ and for any n from $\{0, \ldots, 2p-1\}$, let us set $\lambda_{pn} = -\pi + n\frac{\pi}{p}$ and $A_{pn} = [-\pi + n\frac{\pi}{p}; -\pi + (n+1)\frac{\pi}{p}[$. We remark that $\{A_{pn}; n = 0, \ldots, 2p-1\}$ is a partition of Π. So we have the following property.

Proposition 13. *If U is a unitary operator of H, of associated s.m. $\mathcal{E}$, then*
(i) *for any integer $p \geq 1$, $U_p = \sum_{n=0}^{2p-1} e^{i\lambda_{pn}} \mathcal{E}A_{pn}$ is a unitary operator;*
(ii) *for any X from H, we have $UX = lim_p U_p X = lim_p \sum_{n=0}^{2p-1} e^{i\lambda_{pn}} \mathcal{E}A_{pn} X$.*

2.6. Principal Components Analysis in the frequency domain

The Principal Components Analysis (PCA) in the frequency domain has been first studied by Brillinger [2], and then extended by Boudou and Dauxois [4]. In particular, $\mathbb{Z}$ becomes $\mathbb{Z}^k$, and $\mathbb{C}^k$ becomes a separable $\mathbb{C}-$Hilbert space H. We present here this analysis.

Definition 10. A $H-$stationary series $(X_n)_{n\in\mathbb{Z}^k}$ is a family of elements of $L^2_H(\Omega, \mathcal{A}, P)$ such that $\int X_{n_1} \otimes X_{n_2} \mathrm{d}P = \int X_{n_1-n_2} \otimes X_0 \mathrm{d}P$, for any pair (n_1, n_2) from $\mathbb{Z}^k$.

The PCA in the frequency domain aims to extract, from a $H-$stationary series $(X_n)_{n\in\mathbb{Z}^k}$, a $p-$dimensional summary $(X'_n)_{n\in\mathbb{Z}^k}$, which must be a $\mathbb{C}^p-$stationary series, stationarily correlated with $(X_n)_{n\in\mathbb{Z}^k}$, that is to say, such that $\int X_{n_1} \otimes X'_{n_2} \mathrm{d}P = \int X_{n_1-n_2} \otimes X'_0 \mathrm{d}P$, for any pair (n_1, n_2) from $\mathbb{Z}^k$.

Let us denote by P_p the projector from $L^2_H(\Omega)$ on $\overline{\mathrm{vect}}\{K \circ X'_n; n \in \mathbb{Z}^k, K \in \sigma_2(\mathbb{C}^p, H)\}$. We can prove that $(P_p X_n)_{n\in\mathbb{Z}^k}$ is a $H-$stationary series, stationarily correlated with $(X_n)_{n\in\mathbb{Z}^k}$. Consequently, the quantity $\|X_n - P_p X_n\|$ does not depend on n, and therefore we choose it as a measure of the quality of the $p-$dimensional summary $(X'_n)_{n\in\mathbb{Z}^k}$, of $(X_n)_{n\in\mathbb{Z}^k}$. Of course, among all the possible $p-$dimensional summaries, we retain the one which is the most efficient, according to the following definition.

Definition 11. We name the first p steps of the PCA in the frequency domain of a $H-$stationary series $(X_n)_{n\in\mathbb{Z}^k}$ the search of a $\mathbb{C}^p-$stationary series $(X'_n)_{n\in\mathbb{Z}^k}$, stationarily correlated with $(X_n)_{n\in\mathbb{Z}^k}$, such that $\|X_0 - P_p X_0\|$, where P_p is the projector on $\overline{\mathrm{vect}}\{K \circ X'_n; n \in \mathbb{Z}^k, K \in \sigma_2(\mathbb{C}^p, H)\}$, is as small as possible.

The first p steps provide then a $\mathbb{C}^p-$stationary series $(X'_n)_{n\in\mathbb{Z}^k}$, and a $H-$stationary series $(X''_n)_{n\in\mathbb{Z}^k} = (P_p X_n)_{n\in\mathbb{Z}^k}$. The series $(X'_n)_{n\in\mathbb{Z}^k}$ is a $p-$dimensional summary, $(X''_n)_{n\in\mathbb{Z}^k}$ is a reconstruction of the data.

A $H-$stationary series $(X_n)_{n\in\mathbb{Z}^k}$ is stationary and then we can associate with it a r.m. Z, this last is such that $\int ZA \otimes ZB \mathrm{d}P = 0$, for any pair (A, B) of disjoint elements from $\mathcal{B}_{\Pi^k}$. We say that Z is a $H-$r.m. It plays a great role in the achievement of the PCA in the frequency domain. For example, if $\{Z_j; j \in J\}$ is a finite family of elements from $L^2_H(\mathcal{A})$ such that $\int Z_j \otimes Z_{j'} \mathrm{d}P = 0$, for any pair (j, j') of distinct elements from J, and if $\{\lambda_j, j \in J\}$ is a finite family of pairwise distinct elements from Π^k, then $(\sum_{j\in J} \mathrm{e}^{i\langle\lambda_j, n\rangle} Z_j)_{n\in\mathbb{Z}^k}$ is a $H-$stationary series of associated $H-$r.m. $Z = \sum_{j\in J} \delta_j(.) Z_j$, where δ_j is the Dirac measure defined on $\mathcal{B}_{\Pi^k}$ and concentrated on λ_j. The PCA in the frequency domain amounts to performing the PCA of each of the random vectors Z_j, hence the name "PCA in the frequency domain".

In the introduction, we have presented the particular case where $k = 1$, and we have formulated it as defined by Brillinger [2]. Generally, this PCA gives better results than the PCA of each of the elements X_n from $L^2_H(\Omega)$. In particular cases, the two analyses can be identical, as in the following example. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two independent $\sigma-$fields of $\mathcal{A}$ and such that $\{A_1 \cap A_2; (A_1, A_2) \in \mathcal{B}_1 \times \mathcal{B}_2\}$ generates $\mathcal{A}$, what is equivalent to $\mathcal{A} = T^{-1}(\mathcal{B}_1 \otimes \mathcal{B}_2)$, where T is the mapping $\omega \in \Omega \mapsto (\omega, \omega) \in \Omega \times \Omega$. Let X be an element from $L^2_H(\Omega, \mathcal{B}_1, P)$ and $(y_n)_{n\in\mathbb{Z}^k}$ a stationary series taking values in $L^2(\Omega, \mathcal{B}_2, P)$. We have the following.

Proposition 14. *The sequence $(y_n X)_{n\in\mathbb{Z}^k}$ is a $H-$stationary series of elements from $L^2_H(\Omega, \mathcal{A}, P)$.*

Moreover, if X' and X'' denote elements of respectively $L^2_{\mathbb{C}^p}(\Omega, \mathcal{B}_1, P)$ and $L^2_H(\Omega, \mathcal{B}_1, P)$, which result from the p first steps of the PCA of X, then what precedes can be completed by the following.

Proposition 15. *With the above notation, we can state:*
(i) *$y_n X'$ and $y_n X''$ are respectively elements from $L^2_{\mathbb{C}^p}(\Omega, \mathcal{A}, P)$ and from $L^2_H(\Omega, \mathcal{A}, P)$, resulting from the p first steps of the PCA of $y_n X$, element from $L^2_H(\Omega, \mathcal{A}, P)$.*
(ii) *$(y_n X')_{n\in\mathbb{Z}^k}$ and $(y_n X'')_{n\in\mathbb{Z}^k}$ are respectively the $\mathbb{C}^p-$stationary and $H-$stationary series which result from the p first steps of the PCA in the frequency domain of the $H-$stationary series $(y_n X)_{n\in\mathbb{Z}^k}$.*

3. Conjugate of a spectral measure

In this section we define what is natural to name the conjugate of a spectral measure. We will need this notion in Section 7.4. If K is a bounded endomorphism of $L^2(\Omega)$, then it is the same for $\gamma \circ K \circ \gamma$, and we get the following property.

Proposition 16. *If K is a bounded endomorphism of $L^2(\Omega)$, then $(\gamma \circ K \circ \gamma)^* = \gamma \circ K^* \circ \gamma$.*

Proof: It comes from the following equalities:

$$\langle (\gamma K \gamma)^* u, v\rangle = \langle u, \gamma K \gamma v\rangle = \langle K\gamma v, \gamma u\rangle = \langle \gamma v, K^* \gamma u\rangle = \langle (\gamma K^* \gamma)u, v\rangle.$$

□

Then it is easy to verify that, if K is a projector (resp. a unitary operator of $L^2(\Omega)$), then $\gamma K \gamma$ is a projector (resp. a unitary operator of $L^2(\Omega)$), and that, if $\mathcal{E}$ is a s.m. on $\mathcal{B}_{\Pi^k}$ for $L^2(\Omega)$, then the same happens for the mapping $A \in \mathcal{B}_{\Pi^k} \mapsto \gamma \circ \mathcal{E}A \circ \gamma \in \mathcal{P}(L^2(\Omega))$. This lets us define the conjugate of a s.m.

Definition 12. We name conjugate of $\mathcal{E}$, s.m. on $\mathcal{B}_{\Pi^k}$ for $L^2(\Omega)$, the s.m. defined by $A \in \mathcal{B}_{\Pi^k} \mapsto \gamma \circ \mathcal{E}A \circ \gamma \in \mathcal{P}(L^2(\Omega))$.

The following property defines the unitary operator associated with the conjugate of a s.m.

Proposition 17. *If U is a unitary operator of associated s.m. $\mathcal{E}$, then the unitary operator $\gamma U \gamma$ has as associated s.m. $W_\Pi \mathcal{E}_c$, where $\mathcal{E}_c$ is the conjugate s.m. of $\mathcal{E}$.*

Proof: Let V be the unitary operator of associated s.m. $W_\Pi \mathcal{E}_c$, where $\mathcal{E}_c$ is the conjugate of $\mathcal{E}$. From the recalls of Section 2, we can write

$$VX = \int \mathrm{e}^{i.1}\mathrm{d}Z^X_{W_\Pi \mathcal{E}_c} = \int \mathrm{e}^{i.1}\mathrm{d}W_\Pi(Z^X_{\mathcal{E}_c}) = \int \mathrm{e}^{i.1} \circ W_\Pi \mathrm{d}Z^X_{\mathcal{E}_c} = \int \mathrm{e}^{i.-1}\mathrm{d}Z^X_{\mathcal{E}_c}, \tag{2}$$

and as $Z^X_{\mathcal{E}_c} = \gamma \circ Z^{\gamma X}_{\mathcal{E}}$, (2) can be completed by

$$VX = \int \mathrm{e}^{i.-1}\mathrm{d}\gamma \circ Z^X_{\mathcal{E}} = \gamma\Big(\int \mathrm{e}^{i.1}\mathrm{d}Z^{\gamma X}_{\mathcal{E}}\Big) = \gamma U \gamma X, \tag{3}$$

this for any X from $L^2(\Omega)$, so the property stands. □

We can now generalize this property.

Proposition 18. *If $\{U_n; n \in \mathbb{Z}^k\}$ is the group of the unitary operators deduced from $\mathcal{E}$, s.m. on $\mathcal{B}_{\Pi^k}$ for $L^2(\Omega)$, then $\{\gamma U_n \gamma; n \in \mathbb{Z}^k\}$ is the group of the unitary operators of $L^2(\Omega)$ deduced from the s.m. $W_{\Pi^k}\mathcal{E}_c$, where $\mathcal{E}_c$ is the conjugate s.m. of $\mathcal{E}$.*

Proof: Let us denote by $\{V_n; n \in \mathbb{Z}^k\}$ the group of the unitary operators of $L^2(\Omega)$ deduced from $W_{\Pi^k}\mathcal{E}_c$. If $(n_1, \ldots, n_k)$ is an element from $\mathbb{Z}^k$, we have $V_{(n_1,\ldots,n_k)} = \prod_{j=1}^k V_j^{n_j}$, where V_j is the unitary operator of associated s.m. $P_j W_{\Pi^k}\mathcal{E}_c = W_\Pi P_j \mathcal{E}_c$, for $j \in \{1, \ldots, k\}$. Let U_j be the unitary operator of associated spectral measure $P_j\mathcal{E}$. From what precedes, the unitary operator $\gamma U_j \gamma$ has as associated s.m. $W_\Pi(P_j\mathcal{E})_c$ (where $(P_j\mathcal{E})_c$ is the conjugate measure of $P_j\mathcal{E}$), but as $(P_j\mathcal{E})_c = P_j\mathcal{E}_c$ ($(P_j\mathcal{E})_c A = \gamma \circ (P_j\mathcal{E}A) \circ \gamma = \mathcal{E}_c P_j^{-1} A = P_j\mathcal{E}_c A$, for any A from $\mathcal{B}_\Pi$), the s.m. associated with $\gamma U_j \gamma$ is $W_\Pi P_j \mathcal{E}_c$.

So $V_j = \gamma U_j \gamma$, and we can write $V_{(n_1,\cdots,n_k)} = \prod_{j=1}^k V_j^{n_j} = \prod_{j=1}^k (\gamma U_j \gamma)^{n_j} = \prod_{j=1}^k (\gamma U_j^{n_j} \gamma) = \gamma \prod_{j=1}^k U_j^{n_j} \gamma = \gamma U_n \gamma$.

As a conclusion, $\{\gamma U_n \gamma; n \in \mathbb{Z}^k\}$ is the group of the unitary operators deduced from the s.m. $W_{\Pi^k}\mathcal{E}_c$. □

4. Isometry from $L^2_{L^2(\Omega)}(\Lambda)$ on $L^2_{L^2(\Lambda)}(\Omega)$

The aim of this section is to recall the definition of a well-known isometry (cf. Schaefer [15]) between the $\mathbb{C}-$Hilbert spaces $L^2_{L^2(\Omega)}(\Lambda)$ and $L^2_{L^2(\Lambda)}(\Omega)$, and to obtain new properties which will deserve the conversion cyclostationarity-stationarity.

For any (y,h) from $L^2(\Omega)\times L^2(\Lambda)$, we denote by yh (resp. hy) the mapping $\omega\in\Omega\mapsto y(\omega)h\in L^2(\Lambda)$ (resp. $\lambda\in\Lambda\mapsto h(\lambda)y\in L^2(\Omega)$), which is measurable and of $P-$integrable (resp. $\eta-$integrable) square norm.

If X is an element from $L^2_{L^2(\Lambda)}(\Omega)$ and y an element from $L^2(\Omega)$, then $y(.)X(.)$ is measurable and of $P-$integrable norm. We can then consider $\int yX\mathrm{d}P$, which is an element from $L^2(\Lambda)$, such that $\langle\int yX\mathrm{d}P,h\rangle=\int y(\omega)\langle X(\omega),h\rangle\mathrm{d}P(\omega)$, for any h from $L^2(\Lambda)$. So we can enunciate the following properties.

Proposition 19. *If X is an element from $L^2_{L^2(\Lambda)}(\Omega)$, then the mapping $\widetilde{X}:y\in L^2(\Omega)\mapsto\int yXdP\in L^2(\Lambda)$ is an Hilbert-Schmidt operator. The mapping $:X\in L^2_{L^2(\Lambda)}(\Omega)\mapsto\widetilde{X}\in\sigma_2(L^2(\Omega),L^2(\Lambda))$ is an isometry.*

Especially, taking into account our convention of writing, it is easy to verify that $\widetilde{yh}=(\gamma y)\otimes h$, for any (y,h) from $L^2(\Omega)\times L^2(\Lambda)$. If K is a bounded endomorphism of $L^2(\Lambda)$ and if X is an element from $L^2_{L^2(\Lambda)}(\Omega)$, then $K\circ X$ is also an element from $L^2_{L^2(\Lambda)}(\Omega)$, and we can verify that $\widetilde{K\circ X}=K\circ\widetilde{X}$. Indeed,

$$\langle\widetilde{K\circ X}y,h\rangle=\langle\int KX(\omega)y(\omega)\mathrm{d}P(\omega),h\rangle=\int\langle X(\omega)y(\omega),K^*h\rangle\mathrm{d}P(\omega)=\langle\widetilde{X}y,K^*h\rangle=\langle K\circ\widetilde{X}y,h\rangle.$$

Let us now examine another expression of a covariance operator.

Proposition 20. *For any pair (X,X') of elements of $L^2_{L^2(\Lambda)}(\Omega)$, we have $\int X(\omega)\otimes X'(\omega)dP(\omega)=\widetilde{X'}\circ\widetilde{X}^*$.*

Proof: The mapping $\langle X'(.),K(X(.))\rangle$, and so the mapping $\langle X(.)\otimes X'(.),K\rangle_{\sigma_2}$, is measurable, this for any K from $\sigma_2(L^2(\Lambda))$. Therefore the mapping $\omega\in\Omega\mapsto X(\omega)\otimes X'(\omega)\in\sigma_2(L^2(\Lambda))$ is measurable, as $\sigma_2(L^2(\Lambda))$ is separable.

Moreover, $\int\|X(\omega)\otimes X'(\omega)\|_{\sigma_2}\mathrm{d}P(\omega)=\int\|X(\omega)\|\|X'(\omega)\|\mathrm{d}P(\omega)\leq\|X\|\|X'\|$. So the mapping $\omega\in\Omega\mapsto X(\omega)\otimes X'(\omega)\in\sigma_2$ is measurable and of $P-$integrable norm. We can then consider the element $\int X(\omega)\otimes X'(\omega)\mathrm{d}P(\omega)$ of $\sigma_2(L^2(\Lambda))$. From the previous remark, we can write $\langle\int X(\omega)\otimes X'(\omega)\mathrm{d}P(\omega),K\rangle_{\sigma_2}=\int\langle X(\omega)\otimes X'(\omega),K\rangle\mathrm{d}P(\omega)=\int\langle X'(\omega),KX(\omega)\rangle\mathrm{d}P(\omega)=\langle X',K\circ X\rangle=\langle\widetilde{X'},K\circ\widetilde{X}\rangle=\langle\widetilde{X'}\circ\widetilde{X}^*,K\rangle$, this for any K of σ_2, hence the property. □

Remark 1. A family $(X_n)_{n\in\mathbb{Z}^k}$ of elements of $L^2_{L^2(\Lambda)}(\Omega)$ is $L^2(\Lambda)-$stationary when, for any pair (n,m) from $\mathbb{Z}^k$, $\widetilde{X_n}\circ\widetilde{X_m}^*=\int X_m(\omega)\otimes X_n(\omega)\mathrm{d}P(\omega)=\int X_0(\omega)\otimes X_{n-m}(\omega)\mathrm{d}P(\omega)=\widetilde{X_{n-m}}\circ\widetilde{X_0}^*$.

Propositions 19 and 20 regard the $\mathbb{C}-$Hilbert spaces $L^2_{L^2(\Lambda)}(\Omega)$ and $\sigma_2(L^2(\Omega),L^2(\Lambda))$. Exchanging the roles of $(\Omega,\mathcal{A},P)$ and of (Λ,ξ,η), we get the following dual properties.

Proposition 21.
(i) *For any Y of $L^2_{L^2(\Omega)}(\Lambda)$, the mapping $\widetilde{Y}:h\in L^2(\Lambda)\mapsto\int h(\lambda)Y(\lambda)d\eta(\lambda)\in L^2(\Omega)$ is an Hilbert-Schmidt operator;*
(ii) *the mapping $Y\in L^2_{L^2(\Omega)}(\Lambda)\mapsto\widetilde{Y}\in\sigma_2(L^2(\Lambda),L^2(\Omega))$ is an isometry;*
(iii) *for any (y,h) from $L^2(\Omega)\times L^2(\Lambda)$, we have $\widetilde{hy}=(\Gamma h)\otimes y$.*

For any pair $((y,h),(y',h'))$ of elements of $L^2(\Omega)\times L^2(\Lambda)$, we have $\langle hy,h'y'\rangle=\langle\Gamma h\otimes y,\Gamma h'\otimes y'\rangle=\langle h,h'\rangle\langle y,y'\rangle=\langle\gamma y\otimes h,\gamma y'\otimes h'\rangle=\langle yh,y'h'\rangle$. So there exists an isometry $\mathcal{I}$, and only one, from $\overline{\mathrm{vect}}\{hy;(y,h)\in L^2(\Omega)\times L^2(\Lambda)\}$ on $\overline{\mathrm{vect}}\{yh;(y,h)\in L^2(\Omega)\times L^2(\Lambda)\}$ such that $\mathcal{I}(hy)=yh$, for any (y,h) from $L^2(\Omega)\times L^2(\Lambda)$. As $\overline{\mathrm{vect}}\{hy;(y,h)\in L^2(\Omega)\times L^2(\Lambda)\}=L^2(\Omega)\times L^2(\Lambda)\}=L^2_{L^2(\Omega)}(\Lambda)$ and as $\overline{\mathrm{vect}}\{yh;(y,h)\in L^2(\Omega)\times L^2(\Lambda)\}=L^2_{L^2(\Lambda)}(\Omega)$, what precedes can be expressed as following.

Proposition 22. *There exists an isometry $\mathcal{I}$, and only one, from $L^2_{L^2(\Omega)}(\Lambda)$ on $L^2_{L^2(\Lambda)}(\Omega)$, such that $\mathcal{I}(hy)=yh$, for any (y,h) of $L^2(\Omega)\times L^2(\Lambda)$.*

Juggling with the various isometries that we have just studied, we have the last result of this section.

Proposition 23. *For any Y of $L^2_{L^2(\Omega)}(\Lambda)$, we have $\gamma \circ \widetilde{Y} \circ \Gamma = \widetilde{\mathcal{I}Y}^*$.*

Proof: For any (y, h) from $L^2(\Omega) \times L^2(\Lambda)$,we can write

$$\langle \gamma \circ \widetilde{Y} \circ \Gamma h, y\rangle = \langle \gamma y, \widetilde{Y} \circ \Gamma h\rangle = \langle \Gamma h \otimes \gamma y, \widetilde{Y}\rangle = \langle h.(\gamma y), Y\rangle$$
$$= \langle (\gamma y).h, \mathcal{I}Y\rangle = \langle y \otimes h, \widetilde{\mathcal{I}Y}\rangle = \langle h, \widetilde{\mathcal{I}Y} y\rangle = \langle \widetilde{\mathcal{I}Y}^* h, y\rangle,$$

so we get as expected: $\gamma \circ \widetilde{Y} \circ \Gamma = \widetilde{\mathcal{I}Y}^*$. □

From now on, $\mathcal{I}$ will keep standing for the same isometry as that we have defined and studied in this section.

5. Family of stationary series pairwise stationarily correlated

In this section, we associate a s.m., on $\mathcal{B}_{\Pi^k}$ for the $\mathbb{C}-$Hilbert space H, with a family of stationary series.

Proposition 24. *If $\{(X^\lambda_m)_{m\in\mathbb{Z}^k}; \lambda \in \Lambda\}$ is a family of stationary series, of elements from H, pairwise stationarily correlated, then there exists a s.m. $\mathcal{E}$, on $\mathcal{B}_{\Pi^k}$ for H, such that if $\{U_n; n \in \mathbb{Z}^k\}$ is the group of the unitary operators of H deduced from $\mathcal{E}$, then $U_m X^\lambda_n = X^\lambda_{m+n}$, this for any (λ, m, n) of $\Lambda \times \mathbb{Z}^k \times \mathbb{Z}^k$.*

Proof: Let P be the projector from H onto $H' = \overline{\text{vect}}\{X^\lambda_n; (\lambda, n) \in \Lambda \times \mathbb{Z}^k\}$ and L the canonical injection $h \in H' \mapsto h \in H$.

Let us consider an element m from $\mathbb{Z}^k$. For any pair $((\lambda_1, n_1); (\lambda_2, n_2))$ of elements from $\Lambda \times \mathbb{Z}^k$, we have $\langle X^{\lambda_1}_{n_1}, X^{\lambda_2}_{n_2}\rangle = \langle X^{\lambda_1}_{n_1+m}, X^{\lambda_2}_{n_2+m}\rangle$. If we notice that $\overline{\text{vect}}\{X^\lambda_{n+m}; (\lambda, n) \in \Lambda \times \mathbb{Z}^k\} = H'$, we can affirm that there exists a unitary operator V_m, and only one, of H', such that $V_m X^\lambda_n = X^\lambda_{n+m}$, for any (λ, n) of $\Lambda \times \mathbb{Z}^k$. It is easy to verify that $V_0 = I_{H'}$. If (p, q) is a pair of elements of $\mathbb{Z}^k$, for any (λ, n) of $\Lambda \times \mathbb{Z}^k$, it comes $V_p V_q X^\lambda_n = V_p X^\lambda_{n+q} = X^\lambda_{n+q+p}$. So, taking into account the property of unicity of V_{p+q}, we have $V_p V_q = V_{p+q}$, and so $V^*_p = V_{-p}$ ($V_{-p} = V_{-p}(V_p V^*_p) = (V_{-p}V_p)V^*_p = V_0 V^*_p = V^*_p$). For any m from $\mathbb{Z}^k$, let us denote $U_m = P_\perp + L \circ U_m \circ L^*$, where $P_\perp$ stands for $I - P$. If we notice that $L^*h = Ph$, for any h from H, that $L^*L = I_{H'}$, $LL^* = P$, $PL = L$, and that $P_\perp L = 0$, it is easy to verify the following assertions:

(i) U_m is a unitary operator of H, for any m from $\mathbb{Z}^k$;

(ii) $U_0 = I_H$;

(iii) $U_p U_q = U_{p+q}$, for any pair (p, q) from $\mathbb{Z}^k$;

(iv) $U^*_p = U_{-p}$, for any p from $\mathbb{Z}^k$.

Obviously, $\{(U_m X)_{m\in\mathbb{Z}^k}; X \in H\}$ is a family of stationary series pairwise stationarily correlated, and if we denote by Z^X the r.m. associated with the stationary series $(U_m X)_{m\in\mathbb{Z}^k}$, $\{Z^X; X \in H\}$ is a family of r.m.'s such that $Z^X \Pi^k = X$, for any X from H and such that, for any pair (X, X') from $H \times H$, the r.m.'s Z^X and $Z^{X'}$ are stationarily correlated. From the recalls of Section 2, we have

(i) for any A from $\mathcal{B}_{\Pi^k}$, the mapping $\mathcal{E}A : X \in H \mapsto Z^X A \in H$ is a projector;

(ii) the mapping $\mathcal{E} : A \in \mathcal{B}_{\Pi^k} \mapsto \mathcal{E}A \in \mathcal{P}(H)$ is a s.m.

Let us denote by $\{W_m; m \in \mathbb{Z}^k\}$ the group of the unitary operators of H deduced from $\mathcal{E}$, $(W_m X)_{m\in\mathbb{Z}^k}$ is a stationary series of associated r.m. $Z^X_{\mathcal{E}}$, but as $Z^X_{\mathcal{E}} = Z^X$ ($Z^X_{\mathcal{E}} A = \mathcal{E}AX = Z^X A$), we have $(W_m X)_{m\in\mathbb{Z}^k} = (U_m X)_{m\in\mathbb{Z}^k}$. So $W_m X = U_m X$, for any X from H, so $W_m = U_m$, for any m from $\mathbb{Z}^k$. We can then affirm that $\{U_m; m \in \mathbb{Z}^k\}$ is the group of the unitary operators of H deduced from the s.m. $\mathcal{E}$, what ends the proof ($U_m X^\lambda_n = X^\lambda_{n+m}$, for any (λ, n, m) of $\Lambda \times \mathbb{Z}^k \times \mathbb{Z}^k$). □

According to Proposition 24, it is natural to consider the notion of s.m. compatible with a family of stationary series.

Definition 13. We say that a s.m. $\mathcal{E}$, on $\mathcal{B}_{\Pi^k}$ for H, is compatible with a family of stationary series pairwise stationarily correlated, $\{(X^\lambda_m)_{m\in\mathbb{Z}^k}; \lambda \in \Lambda\}$ when, denoting by $\{U_m; m \in \mathbb{Z}^k\}$ the group of the unitary operators deduced from $\mathcal{E}$, we have $U_m X^\lambda_n = X^\lambda_{n+m}$, for any (λ, n, m) of $\Lambda \times \mathbb{Z}^k \times \mathbb{Z}^k$.

Remark 2. From Proposition 24, it is clear that with any family of stationary series pairwise stationarily correlated, $\{(X^\lambda_m)_{m\in\mathbb{Z}^k}; \lambda \in \Lambda\}$, we can associate a compatible s.m.

Thanks to Proposition 24, we can link a s.m. with a family of stationary series pairwise stationarily correlated. This is one of the main results of this text, and we can enunciate it as follows.

Theorem 1. *If $\mathcal{E}$ is a s.m., on $\mathcal{B}_{\Pi^k}$ for the $\mathbb{C}-$Hilbert space H, compatible with the family of stationary series pairwise stationarily correlated $\{(X_m^\lambda)_{m\in\mathbb{Z}^k}; \lambda\in\Lambda\}$, then, for any (λ, m) from $\Lambda\times\mathbb{Z}^k$, we have $X_m^\lambda = \int e^{i\langle .,m\rangle} dZ_{\mathcal{E}}^{X_0^\lambda}$.*

Proof: Let us denote by $\{U_m; m\in\mathbb{Z}^k\}$ the group of the unitary operators of H deduced from $\mathcal{E}$. We have $(U_m X_0^\lambda)_{m\in\mathbb{Z}^k} = (X_m^\lambda)_{m\in\mathbb{Z}^k}$. So the stationary series $(X_m^\lambda)_{m\in\mathbb{Z}^k}$ has as associated r.m. $Z_{\mathcal{E}}^{X_0^\lambda}$ (because $\{U_m; m\in\mathbb{Z}^k\}$ is the group of the unitary operators of H deduced from $\mathcal{E}$). For any m from $\mathbb{Z}^k$, we have $X_m^\lambda = \int \mathrm{e}^{i\langle .,m\rangle} \mathrm{d}Z_{\mathcal{E}}^{X_0^\lambda}$, for any λ from Λ. □

6. Ampliation

In this section, for X element from $\mathcal{L}_H^2(\Lambda,\xi,\eta)$, we denote by $\overline{X}$ its coset, which consequently belongs to $L_H^2(\Lambda,\xi,\eta)$. Let K be a bounded endomorphism of H. If X a measurable and of $\eta-$integrable square norm mapping from Λ into H, then it is the same for $K\circ X$. Moreover, if $X = X'$ $\eta-$almost everywhere, then $K\circ X = K\circ X'$ $\eta-$almost everywhere. So we can consider the mapping from $L_H^2(\Lambda)$ into itself, which, with the coset of X associates the coset of $K\circ X$. We name this mapping the ampliation of K and we denote it $\widehat{K}$. Of course, H stands for a separable $\mathbb{C}-$Hilbert space, and $L(H)$ for the Banach space of the continuous bounded linear operators from H into H.

Definition 14. We name ampliation of an element K of $L(H)$, the mapping $\widehat{K} : \overline{X}\in L_H^2(\Lambda)\mapsto \overline{K\circ X}\in L_H^2(\Lambda)$.

Giving an element K of $L(H)$, it is easy to establish the linearity of $\widehat{K}$. Moreover, as $\|\widehat{K\overline{X}}\|^2 = \|\overline{K\circ X}\|^2 = \int\|KX(\lambda)\|^2\mathrm{d}\eta(\lambda) \le \int |||K|||^2\|X(\lambda)\|^2\mathrm{d}\eta(\lambda) = |||K|||^2\|\overline{X}\|^2$, we have the following result.

Proposition 25. *The ampliation $\widehat{K}$ of an element K from $L(H)$ is linear and continuous.*

We let to the reader the exercice of the proof of the following properties.

Proposition 26.
(i) *The ampliation of the identity of H is the identity of $L_H^2(\Lambda)$;*
(ii) *for any pair (K_1, K_2) of elements from $L(H)$, we have $\widehat{K_1\circ K_2} = \widehat{K_1}\circ\widehat{K_2}$.*

The adjoint of the ampliation is the ampliation of the adjoint.

Proposition 27. *For any K from $L(H)$, we have $\widehat{K}^* = \widehat{K^*}$.*

Proof: This results from the equalities:

$$\begin{aligned}\langle\widehat{K^*}\overline{X},\overline{Y}\rangle = \langle\overline{X},\widehat{K}\overline{Y}\rangle = \langle\overline{X},\overline{K\circ Y}\rangle = \int\langle X(\lambda), KY(\lambda)\rangle\mathrm{d}\eta(\lambda)\\ = \int\langle K^*X(\lambda), Y(\lambda)\rangle\mathrm{d}\eta(\lambda) = \langle\overline{K^*\circ X},\overline{Y}\rangle = \langle\widehat{K^*}\overline{X},\overline{Y}\rangle,\end{aligned}$$

for any pair $(\overline{X},\overline{Y})$ of elements from $L_H^2(\Lambda)$. □

We can deduce the following from the two previous propositions.

Proposition 28. *If K is a projector (resp. a unitary operator) of H, then $\widehat{K}$ is a projector (resp. a unitary operator) of $L_H^2(\Lambda)$.*

Now we have the necessary tools in order to define the ampliation of a s.m.

Proposition 29. *If $\mathcal{E}$ is a s.m. on $\mathcal{B}_{\Pi^k}$ for H, then the mapping $A\in\mathcal{B}_{\Pi^k}\mapsto\widehat{\mathcal{E}A}\in\mathcal{P}(L_H^2(\Lambda))$ is a s.m. on $\mathcal{B}_{\Pi^k}$ for $L_H^2(\Lambda)$.*

Proof: The first two axioms of Definition 7 are easy to verify. So it remains to prove that, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements from $\mathcal{B}_{\Pi^k}$ which decreasingly converges to $\emptyset$, then $\lim_n \widehat{\mathcal{E}A_n}\overline{X} = 0$, for any $\overline{X}$ from $L^2_H(\Lambda)$. As $\mathcal{E}$ is a s.m., for any λ from Λ, we have $\lim_n \|\mathcal{E}A_nX(\lambda)\|^2 = 0$. As, on another side, $\|\mathcal{E}A_nX(\lambda)\|^2 \leq \|X(\lambda)\|^2$, the theorem of the dominated convergence lets us get

$$\lim_n \int \|\mathcal{E}A_nX(\lambda)\|^2 \mathrm{d}\eta(\lambda) = \int 0 \mathrm{d}\eta(\lambda) = 0,$$

so $\lim_n \|\widehat{\mathcal{E}A_n}\overline{X}\|^2 = 0$, and we can conclude. □

So we get the definition.

Definition 15. We name ampliation of $\mathcal{E}$, s.m. on $\mathcal{B}_{\Pi^k}$ for H, the s.m. on $\mathcal{B}_{\Pi^k}$ for $L^2_H(\Lambda)$ defined by $\widehat{\mathcal{E}} : A \in \mathcal{B}_{\Pi^k} \mapsto \widehat{\mathcal{E}A} \in \mathcal{P}(L^2_H(\Lambda))$.

Let us now study the s.m. which is associated with $\widehat{U}$, unitary operator of $L^2_H(\Lambda)$, ampliation of the unitary operator U of H.

Proposition 30. *The s.m. associated with the unitary operator $\widehat{U}$, ampliation of the unitary operator U, of associated s.m. $\mathcal{E}$, is $\widehat{\mathcal{E}}$, ampliation of $\mathcal{E}$.*

Proof: Let $\overline{X}$ be an element from $L^2_H(\Lambda)$. From the recalls of Section 2.5, for any λ from Λ, we have

$$\lim_p \| \textstyle\sum_{n=0}^{2p-1} \mathrm{e}^{i\lambda_{pn}} \mathcal{E}A_{pn}X(\lambda) - UX(\lambda)\|^2 = 0.$$

As, for any λ from Λ, $\| \sum_{n=0}^{2p-1} \mathrm{e}^{i\lambda_{pn}} \mathcal{E}A_{pn}X(\lambda) - UX(\lambda)\|^2 \leq 4\|X(\lambda)\|^2$, the theorem of the dominated convergence lets us write

$$\begin{aligned} 0 = \lim_p \int \| \textstyle\sum_{n=0}^{2p-1} \mathrm{e}^{i\lambda_{pn}} \mathcal{E}A_{pn}X(\lambda) - UX(\lambda)\|^2 \mathrm{d}\eta(\lambda) &= \lim_p \| \textstyle\sum_{n=0}^{2p-1} \mathrm{e}^{i\lambda_{pn}} \overline{\mathcal{E}A_{pn} \circ X} - \overline{U \circ X}\|^2 \\ &= \lim_p \| \textstyle\sum_{n=0}^{2p-1} \mathrm{e}^{i\lambda_{pn}} \widehat{\mathcal{E}A_{pn}}\overline{X} - \widehat{U\overline{X}}\|^2. \end{aligned}$$

Hence $\widehat{U\overline{X}} = \lim_p \sum_{n=0}^{2p-1} \mathrm{e}^{i\lambda_{pn}} \widehat{\mathcal{E}}(A_{pn})\overline{X}$, so the property stands. □

This result can be generalized as follows.

Proposition 31. *If $\{U_n; n \in \mathbb{Z}^k\}$ is the group of the unitary operators of H deduced from $\mathcal{E}$, s.m. on $\mathcal{B}_{\Pi^k}$ for H, then $\{\widehat{U}_n; n \in \mathbb{Z}^k\}$ is the group of the unitary operators of $L^2_H(\Lambda)$ deduced from the s.m. $\widehat{\mathcal{E}}$, s.m. on $\mathcal{B}_{\Pi^k}$ for $L^2_H(\Lambda)$, ampliation of $\mathcal{E}$.*

Proof: If U_j, $j \in \{1, \ldots, k\}$, stands for the unitary operator of H, of associated s.m. $P_j\mathcal{E}$, we know that $U_{(n_1,\ldots,n_k)} = \prod_{j=1}^k U_j^{n_j}$, for any $(n_1, \ldots, n_k)$ from $\mathbb{Z}^k$. As U_j is a unitary operator of associated s.m. $P_j\mathcal{E}$, from what precedes, the unitary operator $\widehat{U_j}$ has as associated $\widehat{P_j\mathcal{E}}$, and so $P_j(\widehat{\mathcal{E}})$ (for any A of $\mathcal{B}_\Pi$, $\widehat{P_j\mathcal{E}}A = \widehat{P_j\mathcal{E}A} = \widehat{\mathcal{E}P_j^{-1}A} = \widehat{\mathcal{E}}(P_j^{-1}A) = P_j\widehat{\mathcal{E}}(A)$). The group of unitary operators of $L^2_H(\Lambda)$ deduced from the s.m. $\widehat{\mathcal{E}}$, s.m. on $\mathcal{B}_{\Pi^k}$ for $L^2_H(\Lambda)$, is then $\{\prod_{j=1}^k \widehat{U_j}^{n_j}; (n_1, \ldots, n_k) \in \mathbb{Z}^k\}$ that is $\{\prod_{j=1}^k \widehat{U_j^{n_j}}; (n_1, \ldots, n_k) \in \mathbb{Z}^k\}$, or evenmore $\{\widehat{\prod_{j=1}^k U_j^{n_j}}; (n_1, \ldots, n_k) \in \mathbb{Z}^k\}$. □

7. Cyclostationary random function

This section is the main part of this text. After a sketch of the problem of decomposition of an element from $\mathbb{R}^k$, we give a definition of the cyclostationarity. Then we will show how, from such a function, we can define a $L^2(\Lambda)-$stationary series and hence perform the PCA in the frequency domain. Finally, we will perform, in a certain way, the converse operation: from a $L^2(\Lambda)-$stationary series, we will deduce a cyclostationary r.f. This last operation is useful for the reconstruction of the data.

7.1. Decomposition of an element from $\mathbb{R}^k$

Let us set $\Lambda = [0;\Delta_1[\times\cdots\times[0;\Delta_k[$, where $\Delta_1,\ldots,\Delta_k$ are k elements ($k\geq 1$) of $\mathbb{R}_+^*$. We set $\Delta=(\Delta_1,\ldots,\Delta_k)$. As the topology of $\mathbb{R}^k$ is of countable basis, its Borel $\sigma-$field, $\mathcal{B}_{\mathbb{R}^k}$, is generated by a countable family, and so it is for the $\sigma-$field of subsets of Λ, trace of $\mathcal{B}_{\mathbb{R}^k}$ on Λ, that we will denote ξ. From now on, we denote by η a bounded measure defined on ξ. Therefore, the $\mathbb{C}-$Hilbert space $L^2(\Lambda,\xi,\eta)$ is separable (because ξ is generated by a countable family).

Let us now examine the decomposition of an element from $\mathbb{R}^k$ into the sum of an element from Λ and an element from the subgroup $\Delta_1\mathbb{Z}\times\cdots\times\Delta_k\mathbb{Z}$. For this, if $n=(n_1,\ldots,n_k)$ is an element from $\mathbb{Z}^k$, let us denote by $n\odot\Delta$ the element $(n_1\Delta_1,\ldots,n_k\Delta_k)$ of $\Delta_1\mathbb{Z}\times\cdots\times\Delta_k\mathbb{Z}$. It is easy to verify that $0_{\mathbb{R}^k}\odot\Delta=0_{\mathbb{R}^k}$ and that $n\odot\Delta+m\odot\Delta=(n+m)\odot\Delta$, for any pair (n,m) of elements from $\mathbb{Z}^k$. Let us examine the following property of decomposition.

Proposition 32. *Let $(t_1,\ldots,t_k)$ be an element from $\mathbb{R}^k$. There exists an element from $\Lambda\times\mathbb{Z}^k$, and only one, $((t_1-\Delta_1\left[\frac{t_1}{\Delta_1}\right],\ldots,t_k-\Delta_k\left[\frac{t_k}{\Delta_k}\right]),(\left[\frac{t_1}{\Delta_1}\right],\ldots,\left[\frac{t_k}{\Delta_k}\right]))$, such that $(t_1,\ldots,t_k)=(t_1-\Delta_1\left[\frac{t_1}{\Delta_1}\right],\ldots,t_k-\Delta_k\left[\frac{t_k}{\Delta_k}\right])+(\left[\frac{t_1}{\Delta_1}\right],\ldots,\left[\frac{t_k}{\Delta_k}\right])\odot\Delta$.*

Proof: If $(t_1,\ldots,t_k)$ is an element from $\mathbb{R}^k$, it is clear that $(t_1-\Delta_1\left[\frac{t_1}{\Delta_1}\right],\ldots,t_k-\Delta_k\left[\frac{t_k}{\Delta_k}\right])$ belongs to Λ (as $\left[\frac{t_j}{\Delta_j}\right]\leq\frac{t_j}{\Delta_j}<\left[\frac{t_j}{\Delta_j}\right]+1$, we have $0\leq t_j-\Delta_j\left[\frac{t_j}{\Delta_j}\right]<\Delta_j$, that is $t_j-\Delta_j\left[\frac{t_j}{\Delta_j}\right]\in[0;\Delta_j[$, $j\in\{1,\ldots,k\}$).

So $((t_1-\Delta_1\left[\frac{t_1}{\Delta_1}\right],\ldots,t_k-\Delta_k\left[\frac{t_k}{\Delta_k}\right]),(\left[\frac{t_1}{\Delta_1}\right],\ldots,\left[\frac{t_k}{\Delta_k}\right]))$ is indeed an element from $\Lambda\times\mathbb{Z}^k$, and $(t_1-\Delta_1\left[\frac{t_1}{\Delta_1}\right],\ldots,t_k-\Delta_k\left[\frac{t_k}{\Delta_k}\right])+(\left[\frac{t_1}{\Delta_1}\right],\ldots,\left[\frac{t_k}{\Delta_k}\right])\odot\Delta=(t_1,\ldots,t_k)$.

Now we have to prove the unicity of this decomposition. Let then $((\lambda_1,\ldots,\lambda_k),(n_1,\ldots,n_k))$ be an element from $\Lambda\times\mathbb{Z}^k$ such that $(\lambda_1,\ldots,\lambda_k)+(n_1,\ldots,n_k)\odot\Delta=(t_1,\ldots,t_k)$. For $j\in\{1,\ldots,k\}$, we have then $\lambda_j+n_j\Delta_j=t_j$, so $\frac{\lambda_j}{\Delta_j}+n_j=\frac{t_j}{\Delta_j}$, and $\left[\frac{\lambda_j}{\Delta_j}\right]+n_j=\left[\frac{t_j}{\Delta_j}\right]$. But as $\left[\frac{\lambda_j}{\Delta_j}\right]=0$ (because $0\leq\lambda_j<\Delta_j$), it comes $n_j=\left[\frac{t_j}{\Delta_j}\right]$. We deduce that $(\lambda_j,n_j)=(t_j-\Delta_j\left[\frac{t_j}{\Delta_j}\right],\left[\frac{t_j}{\Delta_j}\right])$, and then that $((\lambda_1,\ldots,\lambda_k),(n_1,\ldots,n_k))=((t_1-\Delta_1\left[\frac{t_1}{\Delta_1}\right],\ldots,t_k-\Delta_k\left[\frac{t_k}{\Delta_k}\right]),(\left[\frac{t_1}{\Delta_1}\right],\ldots,\left[\frac{t_k}{\Delta_k}\right]))$, what ends the proof. □

In the following, for any t from $\mathbb{R}^k$, we denote by (λ_t,n_t) the unique element from $\Lambda\times\mathbb{Z}^k$ such that $t=\lambda_t+n_t\odot\Delta$. We can verify that, for any (t,m) of $\mathbb{R}^k\times\mathbb{Z}^k$, $\lambda_{t+m\odot\Delta}=\lambda_t$, and that $n_{t+m\odot\Delta}=n_t+m$ (because $t+m\odot\Delta=\lambda_t+(n_t+m)\odot\Delta$).

7.2. Definition of the cyclostationarity and first properties

Now we have got the tools and the mathematical frame to define the cyclostationarity.

Definition 16. A family $\{X_t;t\in\mathbb{R}^k\}$ of elements from $L^2(\Omega)$ is a $\Lambda-$cyclostationary r.f. if
(i) the mapping $\lambda\in\Lambda\mapsto X_\lambda\in L^2(\Omega)$ is $\xi-$measurable and of $\eta-$integrable square norm;
(ii) $\langle X_t,X_{t'}\rangle=\langle X_{t+m\odot\Delta},X_{t'+m\odot\Delta}\rangle$, for any (t,t',m) from $\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{Z}^k$.

Point (ii) is tipically used for the definition of the cyclostationarity, in the particular case where $k=1$: $\langle X_t,X_{t'}\rangle=\langle X_{t+m\Delta},X_{t'+m\Delta}\rangle$, for any (t,t',m) from $\mathbb{R}\times\mathbb{R}\times\mathbb{Z}$. Let us notice that Point (i) is little restrictive, it is verified as soon as the mapping $t\in\mathbb{R}^k\mapsto X_t\in L^2(\Omega)$ is continuous. Indeed, in that case, this mapping is $\mathcal{B}_{\mathbb{R}^k}-$measurable (because $\mathcal{B}_{\mathbb{R}^k}$ is the Borel $\sigma-$field), so its restriction to Λ, that is the mapping $\lambda\in\Lambda\mapsto X_t\in L^2(\Omega)$, is $\xi-$measurable (because ξ is the $\sigma-$field trace of $\mathcal{B}_{\mathbb{R}^k}$ on Λ). Moreover, as $[0;\Delta_1]\times\cdots\times[0;\Delta_k]$ is a compact set (because it is closed and bounded), its image by the continuous mapping $\psi:t\in\mathbb{R}^k\mapsto\|X_t\|\in\mathbb{R}$ is a compact subset of $\mathbb{R}$, hence is bounded, as well as $\psi(\Lambda)$, because $\Lambda\subset[0;\Delta_1]\times\cdots\times[0;\Delta_k]$. This lets us write $\int\|X_\lambda\|^2\mathrm{d}\eta(\lambda)<+\infty$.

The following of the subsection is dedicated to the association of a s.m. with a $\Lambda-$cyclostationary r.f., just as we can associate a r.m. with a stationary series.

Proposition 33. *If $\{X_t;t\in\mathbb{R}^k\}$ is a $\Lambda-$cyclostationary r.f., then we can affirm that $\{(X_{\lambda+m\odot\Delta})_{m\in\mathbb{Z}^k};\lambda\in\Lambda\}$ is a family of stationary series, pairwise stationarily correlated.*

Proof: This comes from Point (ii) of Definition 16: for any pair $((\lambda,m),(\lambda',m'))$ of elements from $\Lambda\times\mathbb{Z}^k$, we have
$\langle X_{\lambda+(m-m')\odot\Delta},X_{\lambda'+0\odot\Delta}\rangle=\langle X_{\lambda+(m-m')\odot\Delta+m'\odot\Delta},X_{\lambda'+0\odot\Delta+m'\odot\Delta}\rangle=\langle X_{\lambda+(m-m'+m')\odot\Delta},X_{\lambda'+(0+m')\odot\Delta}\rangle=\langle X_{\lambda+m\odot\Delta},X_{\lambda'+m'\odot\Delta}\rangle$.
□

Section 5 gives us tools for considering a s.m. compatible with a $\Lambda-$cyclostationary r.f.

Definition 17. We say that $\mathcal{E}$, s.m. on $\mathcal{B}_{\Pi^k}$ for $L^2(\Omega)$, is compatible with the $\Lambda-$cyclostationary r.f. $\{X_t; t \in \mathbb{R}^k\}$, when $\mathcal{E}$ is compatible with the family of stationary series pairwise stationarily correlated $\{(X_{\lambda+m\odot\Delta})_{m\in\mathbb{Z}^k}; \lambda \in \Lambda\}$.

Such a s.m. exists, as $\{(X_{\lambda+m\odot\Delta})_{m\in\mathbb{Z}^k}; \lambda \in \Lambda\}$ is a family of stationary series pairwise stationarily correlated. Then we can get the following property.

Proposition 34. *If $\mathcal{E}$ is a s.m. on $\mathcal{B}_{\Pi^k}$ for $L^2(\Omega)$, compatible with the $\Lambda-$cyclostationary r.f. $\{X_t; t \in \mathbb{R}^k\}$, then for any t from $\mathbb{R}^k$, we have $X_t = \int e^{i\langle .,n_t\rangle} dZ_{\mathcal{E}}^{X_{\lambda_t}}$.*

Proof: From Proposition 33, $\{(X_{\lambda+m\odot\Delta})_{m\in\mathbb{Z}^k}; \lambda \in \Lambda\}$ is a family of stationary series, pairwise stationarily correlated, and if $\mathcal{E}$ is a s.m. compatible with the $\Lambda-$cyclostationary r.f. $\{X_t; t \in \mathbb{R}^k\}$, then $\mathcal{E}$ is compatible with the family of stationary series, pairwise stationarily correlated $\{(X_{\lambda+m\odot\Delta})_{m\in\mathbb{Z}^k}; \lambda \in \Lambda\}$. Theorem 1 lets us write $X_{\lambda+m\odot\Delta} = \int \mathrm{e}^{i\langle .,m\rangle} \mathrm{d}Z_{\mathcal{E}}^{X_{\lambda+0\odot\Delta}}$, this for any (λ, m) from $\Lambda\times\mathbb{Z}^k$. Let t be an element from $\mathbb{R}^k$, as $(\lambda_t, n_t) \in \Lambda\times\mathbb{Z}^k$ and as $t = \lambda_t + n_t \odot \Delta$, we have $X_t = X_{\lambda_t+n_t\odot\Delta} = \int \mathrm{e}^{i\langle .,n_t\rangle} \mathrm{d}Z_{\mathcal{E}}^{X_{\lambda_t}}$. □

7.3. Construction of a stationary series from a cyclostationary function

The following result, which is the major result of this text, shows how we use the ampliation to obtain a stationary series from a cyclostationary function. The PCA in the frequency domain of the resulting stationary series provides the spectral analysis of the associated cyclostationary function.

Theorem 2. *If $\mathcal{E}$ is a s.m., on $\mathcal{B}_{\Pi^k}$ for $L^2(\Omega)$, compatible with the $\Lambda-$cyclostationary r.f. $\{X_t; t \in \mathbb{R}^k\}$, we can affirm that*

(i) *for any m from $\mathbb{Z}^k$, the mapping $\lambda \in \Lambda \mapsto X_{\lambda+m\odot\Delta} \in L^2(\Omega)$ is a representative of an element X'_m from $L^2_{L^2(\Omega)}(\Lambda)$;*

(ii) *$(X'_m)_{m\in\mathbb{Z}^k}$ is a stationary series of elements from $L^2_{L^2(\Omega)}(\Lambda)$;*

(iii) *if $Z_{X'}$ is the r.m. associated with the stationary series $(X'_m)_{m\in\mathbb{Z}^k}$, for any A from $\mathcal{B}_{\Pi^k}$, $Z_{X'}A$, element from $L^2_{L^2(\Omega)}(\Lambda)$, has as representative the mapping $\lambda \in \Lambda \mapsto \mathcal{E}AX_\lambda \in L^2(\Omega)$;*

(iv) *$(\mathcal{I}X'_m)_{m\in\mathbb{Z}^k}$ is a $L^2(\Lambda)-$stationary series;*

(v) *if Z is the r.m. associated with $(\mathcal{I}X'_m)_{m\in\mathbb{Z}^k}$, for any A from $\mathcal{B}_{\Pi^k}$, we have $\widetilde{ZA}^* = \gamma \circ \mathcal{E}A \circ \widetilde{X'_0} \circ \Gamma$.*

Proof: When $\mathcal{E}$ is a s.m. compatible with the $\Lambda-$cyclostationary r.f. $\{X_t; t \in \mathbb{R}^k\}$, for any t from $\mathbb{R}^k$, we have

$$X_t = \int \mathrm{e}^{i\langle .,n_t\rangle} \mathrm{d}Z_{\mathcal{E}}^{X_{\lambda_t}} \text{ (cf. Proposition 34).} \tag{4}$$

Let (λ, m) be an element from $\Lambda \times \mathbb{Z}^k$. The unicity of the decomposition of an element of $\mathbb{R}^k$ gives: $n_{\lambda+m\odot\Delta} = m$ and $\lambda_{\lambda+m\odot\Delta} = \lambda$. From (4), we have $X_{\lambda+m\odot\Delta} = \int \mathrm{e}^{i\langle .,m\rangle} \mathrm{d}Z_{\mathcal{E}}^{X_\lambda}$. This means that the stationary series $(X_{\lambda+m\odot\Delta})_{m\in\mathbb{Z}^k}$ has as associated r.m. $Z_{\mathcal{E}}^{X_\lambda}$.

Let us denote $\{U_m; m \in \mathbb{Z}^k\}$ the group of the unitary operators of $L^2(\Omega)$ deduced from the s.m. $\mathcal{E}$. From the recalls of Section 2, $(U_m X_\lambda)_{m\in\mathbb{Z}^k}$ is a stationary series of associated r.m. $Z_{\mathcal{E}}^{X_\lambda}$, and so the stationary series $(U_m X_\lambda)_{m\in\mathbb{Z}^k}$ and $(X_{\lambda+m\odot\Delta})_{m\in\mathbb{Z}^k}$ having the same associated r.m., are equal. Then $U_m X_\lambda = X_{\lambda+m\odot\Delta}$, for any (λ, m) of $\Lambda \times \mathbb{Z}^k$.

From Section 6, $\{\widehat{U_m}; m \in \mathbb{Z}^k\}$ is the group of the unitary operators of $L^2_{L^2(\Omega)}(\Lambda)$, deduced from the s.m. $\widehat{\mathcal{E}}$, ampliation of $\mathcal{E}$.

If we denote X'_0 the coset of $\lambda \in \Lambda \mapsto X_\lambda \in L^2(\Omega)$, $\widehat{U_m}X'_0$ is an element from $L^2_{L^2(\Omega)}(\Lambda)$, which has as representative the mapping $U_m \circ (\lambda \in \Lambda \mapsto X_\lambda \in L^2(\Omega))$, that is to say the mapping $\lambda \in \Lambda \mapsto U_m X_\lambda \in L^2(\Omega)$, or evenmore $\lambda \in \Lambda \mapsto X_{\lambda+m\odot\Delta} \in L^2(\Omega)$),

hence Point (i). Point (ii) comes from $\widehat{U_m}X'_0 = X'_m$. As $(X'_m)_{m\in\mathbb{Z}^k} = (\widehat{U_m}X'_0)_{m\in\mathbb{Z}^k}$, we have $Z_{X'} = Z_{\widehat{\mathcal{E}}}^{X'_0}$.

For any A of $\mathcal{B}_{\Pi^k}$, $Z_{X'_0}A = \widehat{\mathcal{E}}AX'_0 = \widehat{\mathcal{E}A}X'_0$ has as representative the mapping $\mathcal{E}A \circ (\lambda \in \Lambda \mapsto X_\lambda \in L^2(\Omega))$, that is the mapping $\lambda \in \Lambda \mapsto \mathcal{E}AX_\lambda \in L^2(\Omega)$, hence Point (iii).

Let m be an element from $\mathbb{Z}^k$, for any (h, y) from $L^2(\Lambda) \times L^2(\Omega)$, we have

$\langle \widetilde{X'_m}(h), y\rangle = \langle \int h(\lambda) U_m X_\lambda \mathrm{d}\eta(\lambda), y\rangle = \int h(\lambda)\langle X_\lambda, U_{-m}y\rangle \mathrm{d}\eta(\lambda) = \langle \int h(\lambda) X_\lambda \mathrm{d}\eta(\lambda), U_{-m}y\rangle = \langle \widetilde{X'_0}h, U_{-m}y\rangle = \langle U_m \circ \widetilde{X'_0}h, y\rangle,$

so $\widetilde{X'_m} = U_m \circ \widetilde{X'_0}$.

From Section 4, we can write:

$$\widetilde{\mathcal{I}X'_m}^* = \gamma \circ \widetilde{X'_m} \circ \Gamma = \gamma \circ U_m \circ \widetilde{X'_0} \circ \Gamma.$$

For any pair (n, m) from $\mathbb{Z}^k$, it comes

$$\widetilde{\mathcal{I}X'_n}\widetilde{\mathcal{I}X'_m}^* = (\gamma \circ U_n \circ \widetilde{X'_0} \circ \Gamma)^* \gamma \circ U_m \circ \widetilde{X'_0} \circ \Gamma = \Gamma \widetilde{X'_0}^* U_{-n}\gamma\gamma U_m \widetilde{X}^* \Gamma = \Gamma \widetilde{X'_0}^* U_{m-n} \widetilde{X'_0}\Gamma = \widetilde{\mathcal{I}X'_{m-n}}\widetilde{\mathcal{I}X'_0}^*,$$

what lets us affirm that $(\mathcal{I}X'_m)_{m\in\mathbb{Z}^k}$ is a $L^2(\Lambda)-$stationary series. From recalls of Section 2, $\mathcal{I} \circ Z_{X'}$ is a r.m. and

$$\int \mathrm{e}^{\langle \cdot, m\rangle} \mathrm{d}\mathcal{I} \circ Z_{X'} = \mathcal{I} \int \mathrm{e}^{\langle \cdot, m\rangle} \mathrm{d}Z_{X'} = \mathcal{I}X'_m,$$

this for any m from $\mathbb{Z}^k$. We deduce that $Z = \mathcal{I} \circ Z_{X'}$. Let us consider an element A from $\mathcal{B}_{\Pi^k}$, we have, for any (h, y) from $L^2(\Lambda) \times L^2(\Omega)$,

$\langle \widetilde{Z_{X'}A}h, y\rangle = \langle \int h(\lambda)\mathcal{E}AX_\lambda \mathrm{d}\eta(\lambda), y\rangle = \int h(\lambda)\langle X_\lambda, \mathcal{E}Ay\rangle \mathrm{d}\eta(\lambda) = \langle \int h(\lambda)X_\lambda \mathrm{d}\eta(\lambda), \mathcal{E}Ay\rangle = \langle \widetilde{X'_0}h, \mathcal{E}Ay\rangle = \langle \mathcal{E}A \circ \widetilde{X'_0}h, y\rangle$,

so $\widetilde{Z_{X'}A} = \mathcal{E}A \circ \widetilde{X'_0}$. Considering that $ZA = \mathcal{I}(Z_{X'}A)$, Proposition 23 lets us write $\widetilde{ZA}^* = \widetilde{\mathcal{I}(Z_{X'}A)}^* = \gamma \circ \widetilde{Z_{X'}A} \circ \Gamma = \gamma \circ \mathcal{E}A \circ \widetilde{X'_0} \circ \Gamma$, what ends the proof. □

We can then define the $L^2(\Lambda)-$stationary series deduced from a $\Lambda-$cyclostationary r.f.

Definition 18. We name $L^2(\Lambda)-$stationary series deduced from the $\Lambda-$cyclostationary r.f. $\{X_t; t \in \mathbb{R}^k\}$ the $L^2(\Lambda)-$stationary series $(Y_m)_{m\in\mathbb{Z}^k}$ such that, for any m from $\mathbb{Z}^k$, $\mathcal{I}^{-1}Y_m$, element from $L^2_{L^2(\Omega)}(\Lambda)$ has as representative the mapping $\lambda \in \Lambda \mapsto X_{\lambda+m\odot\Delta} \in L^2(\Omega)$.

Remark 3. Let us consider notation of Theorem 2, and denote by Z_λ the r.m. associated with the stationary series $(X_{\lambda+m\odot\Delta})_{m\in\mathbb{Z}^k}$. We have then $Z_\lambda = Z_{\mathcal{E}}^{X_\lambda}$, and, for any A of $\mathcal{B}_{\Pi^k}$, $Z_\lambda A = \mathcal{E}AX_\lambda$. Point (iii) of Theorem 2 can be expressed in the following way: $Z_{X'}A$ has as representative the mapping $\lambda \in \Lambda \mapsto Z_\lambda A \in L^2(\Omega)$.

Remark 4. If $(Y_m)_{m\in\mathbb{Z}^k}$ is a $L^2(\Lambda)-$stationary series deduced from the $\Lambda-$cyclostationary r.f.'s $\{X^1_t; t \in \mathbb{R}^k\}$ and $\{X^2_t; t \in \mathbb{R}^k\}$, then, for any m from $\mathbb{Z}^k$, $\mathcal{I}^{-1}Y_m$, element from $L^2_{L^2(\Omega)}(\Lambda)$ has as representatives the mappings $\lambda \in \Lambda \mapsto X^1_{\lambda+m\odot\Delta} \in L^2(\Omega)$, and $\lambda \in \Lambda \mapsto X^2_{\lambda+m\odot\Delta} \in L^2(\Omega)$. This implies that $\eta(\{\lambda \in \Lambda; X^1_{\lambda+m\odot\Delta} \neq X^2_{\lambda+m\odot\Delta}\}) = 0$, for any m from $\mathbb{Z}^k$.

Remark 5. Point (v) of Theorem 2 is major. It explicits the r.m. Z associated with the $L^2(\Lambda)-$stationary series $(\mathcal{I}X'_m)_{m\in\mathbb{Z}^k}$. This step is necessary to proceed to the PCA in the frequency domain.

7.4. Cyclostationary random function deduced from a stationary series

In the last main result of this text we show how, considering a $L^2(\Lambda)-$stationary series $(Y_n)_{n\in\mathbb{Z}^k}$, we define a $\Lambda-$cyclostationary r.f. for which the deduced $L^2(\Lambda)-$stationary series is $(Y_n)_{n\in\mathbb{Z}^k}$.

Theorem 3. *Let $(Y_n)_{n\in\mathbb{Z}^k}$ be a $L^2(\Lambda)-$stationary series. We can affirm that $\{(\widetilde{Y_n}^* h)_{n\in\mathbb{Z}^k}; h \in L^2(\Lambda)\}$ is a family of stationary series pairwise stationarily correlated. Moreover, if $\{U_m; m \in \mathbb{Z}^k\}$ is the group of the unitary operators of $L^2(\Omega)$ deduced from $\mathcal{E}$, s.m. on $\mathcal{B}_{\Pi^k}$ for $L^2(\Omega)$, compatible with the family $\{(\widetilde{Y_n}^* h)_{n\in\mathbb{Z}^k}; h \in L^2(\Lambda)\}$, and if X is a representative of $\mathcal{I}^{-1}Y_0$, then $\{\gamma \circ U_{n_t}\gamma(X(\lambda_t)); t \in \mathbb{R}^k\}$ is a $\Lambda-$cyclostationary r.f. for which the deduced $L^2(\Lambda)-$stationary series is $(Y_n)_{n\in\mathbb{Z}^k}$.*

Proof: As, for any pair $((n, h), (n', h'))$ from $\mathbb{Z}^k \times L^2(\Lambda)$, we have $\langle \widetilde{Y_n}^* h, \widetilde{Y_{n'}}^* h'\rangle = \langle \widetilde{Y_{n'}}\widetilde{Y_n}^* h, h'\rangle = \langle \widetilde{Y_0}\widetilde{Y_{n-n'}}^* h, h'\rangle = \langle \widetilde{Y_{n-n'}}^* h, \widetilde{Y_0}^* h'\rangle$. We deduce that $\{(\widetilde{Y_n}^* h)_{n\in\mathbb{Z}^k}; h \in L^2(\Lambda)\}$ is a family of stationary series pairwise stationarily correlated.

Let us denote by $\{U_m; m \in \mathbb{Z}^k\}$ the group of the unitary operators of $L^2(\Omega)$ deduced from a s.m. $\mathcal{E}$, on $\mathcal{B}_{\Pi^k}$ for $L^2(\Omega)$, compatible with the family $\{(\widetilde{Y_n}^* h)_{n\in\mathbb{Z}^k}; h \in L^2(\Lambda)\}$. For any t from $\mathbb{R}^k$, let us set $X_t = \gamma U_{n_t}\gamma X(\lambda_t)$ (we recall that (λ_t, n_t) is the unique element from $\Lambda \times \mathbb{Z}^k$ such that $t = \lambda_t + n_t \odot \Delta$). Let t be an element from Λ, $\lambda_t = t$, and $n_t = 0$ ($(t, 0) \in \Lambda \times \mathbb{Z}^k$ and $t + 0 \odot \Delta = t$), so $X_t = \gamma U_0 \gamma X(t)$, we deduce that the mapping $t \in \Lambda \mapsto X_t \in L^2(\Omega)$, equal to X, is indeed $\xi-$measurable and of $\eta-$integrable square norm. We have then issue (i) of Definition 16. For any (t, m) from $\mathbb{R}^k \times \mathbb{Z}^k$, we have

$$X_{t+m\odot\Delta} = \gamma U_{n_{t+m\odot\Delta}}\gamma X(\lambda_{t+m\odot\Delta}) = \gamma U_{n_t+m}\gamma X(\lambda_t) = \gamma U_m\gamma\gamma U_{n_t}\gamma X(\lambda_t) = \gamma U_m\gamma X_t.$$

So we can write $\langle X_t, X_{t'}\rangle = \langle \gamma U_m\gamma X_t, \gamma U_m\gamma X_{t'}\rangle = \langle X_{t+m\odot\Delta}, X_{t'+m\odot\Delta}\rangle$, this for any (t, t', m) from $\mathbb{R}^k \times \mathbb{R}^k \times \mathbb{Z}^k$, hence issue (ii) of Definition 16 stands. We have just proved that $\{X_t; t \in \mathbb{R}^k\} = \{\gamma U_{n_t}\gamma X(\lambda_t); t \in \mathbb{R}^k\}$ is a $\Lambda-$cyclostationary r.f.

Let us now define the $L^2(\Lambda)-$stationary series deduced from $\{X_t; t \in \mathbb{R}^k\}$. Let us firts notice that $U_m \widetilde{Y_0}^* = \widetilde{Y_m}^*$, for any m from $\mathbb{Z}^k$ (because $\{U_m; m \in \mathbb{Z}^k\}$ is the group of the unitary operators of $L^2(\Omega)$ deduced from $\mathcal{E}$, s.m. compatible with the family $\{(\widetilde{Y_n}^* h)_{n\in\mathbb{Z}^k}; h \in L^2(\Lambda)\}$, and then $U_m \widetilde{Y_0}^* h = \widetilde{Y_{m+0}}^* h = \widetilde{Y_m}^* h$, for any h from $L^2(\Lambda)$). From Section 7.3, we know that

(i) for any m from $\mathbb{Z}^k$, the mapping $\lambda \in \Lambda \mapsto X_{\lambda+m\odot\Delta} \in L^2(\Omega)$ is a representative of an element X'_m from $L^2_{L^2(\Omega)}(\Lambda)$;

(ii) $(\mathcal{I}\widetilde{X'_m})_{m\in\mathbb{Z}^k}$ is the $L^2(\Lambda)-$stationary series deduced from $\{X_t; t \in \mathbb{R}^k\}$.

Let m be an element from $\mathbb{Z}^k$. For any (h, y) of $L^2(\Lambda) \times L^2(\Omega)$, we have

$$\langle \widetilde{X'_m} h, y\rangle = \langle \int h(\lambda) X_{\lambda+m\odot\Delta} \mathrm{d}\eta(\lambda), y\rangle = \int h(\lambda)\langle \gamma U_m \gamma X_\lambda, y\rangle \mathrm{d}\eta(\lambda) = \int h(\lambda)\langle X(\lambda), (\gamma U_m \gamma)^* y\rangle \mathrm{d}\eta(\lambda)$$
$$= \langle \int h(\lambda) X_\lambda \mathrm{d}\eta(\lambda), (\gamma U_m \gamma)^* y\rangle = \langle \widetilde{X'_0} h, (\gamma U_m \gamma)^* y\rangle = \langle \gamma U_m \gamma \widetilde{X'_0} h, y\rangle.$$

So $\widetilde{X'_m} = \gamma U_m \gamma \circ \widetilde{X'_0}$.

Results of Proposition 23 let us complete what precedes by

$$\widetilde{\mathcal{I}X'_m}^* = \gamma \circ \widetilde{X'_m} \circ \Gamma = \gamma \circ \gamma \circ U_m \circ \gamma \circ \widetilde{X'_0} \circ \Gamma = U_m \circ \widetilde{\mathcal{I}X'_0}^*.$$

But X'_0, element from $L^2_{L^2(\Omega)}(\Lambda)$, has as representative the mapping $\lambda \in \Lambda \mapsto X_\lambda \in L^2(\Omega)$, that is the mapping X and then $X'_0 = \mathcal{I}^{-1} Y_0$. So what precedes can be writen $\widetilde{\mathcal{I}X'_m}^* = U_m \circ \widetilde{Y_0}^*$, and as $U_m \circ \widetilde{Y_0}^* = \widetilde{Y_m}^*$, we have $\widetilde{\mathcal{I}X'_m}^* = \widetilde{Y_m}^*$ and $\widetilde{\mathcal{I}X'_m} = \widetilde{Y_m}$, or evenmore $\mathcal{I}X'_m = Y_m$. The series $(Y_m)_{m\in\mathbb{Z}^k} = (\mathcal{I}X'_m)_{m\in\mathbb{Z}^k}$ is indeed the $L^2(\Lambda)-$stationary series deduced from the $\Lambda-$cyclostationary r.f. $\{\gamma U_{n_t} \gamma X(\lambda_t); t \in \mathbb{R}^k\}$. □

7.5. Principal Components Analysis

Let $\{X_t; t \in \mathbb{R}^k\}$ be a $\Lambda-$cyclostationary r.f., and let us denote by $(Y_n^p)_{n\in\mathbb{Z}^k}$ the $L^2(\Lambda)-$stationary series issued from the p first steps of the PCA in the frequency domain of the $L^2(\Lambda)-$stationary series $(Y_n)_{n\in\mathbb{Z}^k}$ deduced from $\{X_t; t \in \mathbb{R}^k\}$. From Section 7.4, we know how to define a $\Lambda-$cyclostationary r.f. $\{X_t^p; t \in \mathbb{R}^k\}$ from which the deduced $L^2(\Lambda)-$stationary series is $(Y_n^p)_{n\in\mathbb{Z}^k}$. As $\mathcal{I}^{-1}Y_n$ (resp. $\mathcal{I}^{-1}Y_n^p$), element from $L^2_{L^2(\Omega)}(\Lambda)$, has as representative the mapping $\lambda \in \Lambda \mapsto X_{\lambda+n\odot\Delta} \in L^2(\Omega)$ (resp. $\lambda \in \Lambda \mapsto X^p_{\lambda+n\odot\Delta} \in L^2(\Omega)$), we can write, for any n from $\mathbb{Z}^k$,

$$\|Y_0 - Y_0^p\|^2_{L^2_{L^2(\Lambda)}(\Omega)} = \|Y_n - Y_n^p\|^2_{L^2_{L^2(\Lambda)}(\Omega)} = \|\mathcal{I}^{-1}Y_n - \mathcal{I}^{-1}Y_n^p\|^2_{L^2_{L^2(\Omega)}(\Lambda)} = \int \|X_{\lambda+n\odot\Delta} - X^p_{\lambda+n\odot\Delta}\|^2_{L^2(\Omega)} \mathrm{d}\eta(\lambda).$$

The quantity $\int \|X_{\lambda+n\odot\Delta} - X^p_{\lambda+n\odot\Delta}\|^2 \mathrm{d}\eta(\lambda)$, which is independent of n, measures the quality of the reconstruction of the data. In Boudou and Viguier-Pla [6], we consider the particular case where $k = 1$, $\Delta = 1$, and $\Omega = \{\omega_1, \ldots, \omega_{10}\}$.

8. Simulation

We consider the very particular case where $k = 1$, $\Lambda = [0; 1[$, η is the Lebesgue measure defined on ξ, $\sigma-$field of the subsets of $[0; 1[$, trace of $\mathcal{B}_\mathbb{R}$ on $[0; 1[$. and $\Omega = \{\omega_1, \ldots, \omega_m\}$, with $P(\omega_i) = \frac{1}{m}$, $i \in \{1, \ldots, m\}$. If we set $y_j = \sqrt{m} 1_{\{\omega_j\}}$, $(y_1, \ldots, y_m)$ is an orthonormal basis on $L^2(\Omega)$.

Let us consider a $[0; 1[-$cyclostationary r.f. $\{X_t; t \in \mathbb{R}\}$. As the mapping $t \in [0; 1[\mapsto X_t \in L^2(\Omega)$ is $\xi-$measurable and of $\eta-$integrable square norm, the mapping $f_j : t \in [0; 1[\mapsto \langle X_t, y_j\rangle \in \mathbb{C}$, that is the mapping $t \in [0; 1[\mapsto \frac{1}{\sqrt{m}} X_t(\omega_j) \in \mathbb{C}$, is $\xi-$measurable and of $\eta-$integrable square norm. Let U be a unitary operator of $L^2(\Omega)$ such that $UX_t = X_{t+1}$, for any real t (we know that at least one of such operators exists). For any real t, we then have $X_t = \sum_{j=1}^m f_j(t - [t]) U^{[t]} y_j$, what lets us get the m temporal trajectories:

$$X_t(\omega_l) = \sqrt{m} \sum_{j=1}^m f_j(t - [t])(B^{[t]})_{lj},$$

where B is the matrix expression of U relatively to the basis $(y_1, \ldots, y_m)$:

$$(B^{[t]})_{lj} = \langle U^{[t]} y_j, y_l\rangle = \sum_{q=1}^m \frac{1}{m} (U^{[t]} y_j)(\omega_q) \overline{y_l(\omega_q)} = \frac{1}{\sqrt{m}} (U^{[t]} y_j)(\omega_l).$$

Let us now examine the expression of the associated stationary series $(\mathcal{I}X'_n)_{n\in\mathbb{Z}}$. For any (t, n) from $[0; 1[\times\mathbb{Z}$, we have $X_{t+n} = \sum_{j=1}^m f_j(t) U^n y_j$. So, for any n from $\mathbb{Z}$, $X'_n = \sum_{j=1}^m f_j(U^n y_j)$, hence $\mathcal{I}X'_n = \sum_{j=1}^m (U^n y_j) f_j$, and then, for any l of $\{1, \ldots, m\}$, $(\mathcal{I}X'_n)(\omega_l) = \sqrt{m} \sum_{j=1}^m (B^n)_{lj} f_j$, expression to be brought together with the trajectory equations.

In order to get the r.m. associated with the series $(\mathcal{I}X'_n)_{n\in\mathbb{Z}}$, we have to consider the s.m. $\mathcal{E}$ associated with U. As $\mathcal{E}$ is a s.m. on $\mathcal{B}_\Pi$ for $L^2(\Omega)$, vector space of dimension m, it is concentrated on a finite number of elements from Π, $\lambda_1,\ldots,\lambda_{m'}$, $m'\leq m$: $\mathcal{E}=\sum_{l=1}^{m'}\delta_{\lambda_l}(.)P_l$, $\{P_l, l=1,\ldots,m'\}$ is a family of projectors such that $\sum_{l=1}^{m'}P_l=I_{L^2(\Omega)}$.

Then we have $U^n=\sum_{l=1}^{m'}\mathrm{e}^{i\lambda_l n}P_l$, for any n from $\mathbb{Z}$. So we can write

$$\mathcal{I}X'_n=\sum_{j=1}^{m}\sum_{l=1}^{m'}\mathrm{e}^{i\lambda_l n}(P_l y_j)f_j=\sum_{l=1}^{m'}\mathrm{e}^{i\lambda_l n}\sum_{j=1}^{m}(P_l y_j)f_j.$$

If we set $Z_l=\sum_{j=1}^{m}(P_l y_j)f_j$, $l=1,\ldots,m'$, we can verify that $\widetilde{Z}_l\widetilde{Z}^*_{l'}=0$, for any pair (l,l') of distinct elements of $\{1,\ldots,m'\}$ and what precedes can be writen

$$\mathcal{I}X'_n=\sum_{l=1}^{m'}\mathrm{e}^{i\lambda_l n}Z_l,$$

for any n from $\mathbb{Z}$.

The r.m. associated with the $L^2([0;1[)-$stationary series $(\mathcal{I}X'_n)_{n\in\mathbb{Z}}$ is then

$$Z=\sum_{l=1}^{m'}\delta_{\lambda_l}(.)Z_l.$$

It is the usual PCA of each component $Z_1,\ldots,Z_{m'}$, which gives the PCA in the frequency domain of $(\mathcal{I}X'_n)_{n\in\mathbb{Z}}$.

We consider the particular case where $m=4$, and we set

$$P_1=\tfrac{1}{\sqrt{2}}(y_1+iy_4)\otimes\tfrac{1}{\sqrt{2}}(y_1+iy_4);\ P_2=y_2\otimes y_2+y_3\otimes y_3;\ P_3=\tfrac{1}{\sqrt{2}}(y_1-iy_4)\otimes\tfrac{1}{\sqrt{2}}(y_1-iy_4);$$

$$\lambda_1=-\lambda;\ \lambda_2=0;\ \lambda_3=\lambda;\ m'=3;$$

$$B=\tfrac{1}{2}\mathrm{e}^{-i\lambda}\begin{pmatrix}1\\0\\0\\i\end{pmatrix}\begin{pmatrix}1&0&0&-i\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}+\tfrac{1}{2}\mathrm{e}^{i\lambda}\begin{pmatrix}1\\0\\0\\-i\end{pmatrix}\begin{pmatrix}1&0&0&i\end{pmatrix}=\begin{pmatrix}\cos\lambda&0&0&-\sin\lambda\\0&1&0&0\\0&0&1&0\\\sin\lambda&0&0&\cos\lambda\end{pmatrix}.$$

If we choose $f_1(t)=\cos(2\pi t)(\cos(\lambda t)-\sin(\lambda t))$, $f_2(t)=\cos(2\pi t)$, $f_3(t)=-\cos(2\pi t)$, $f_4(t)=\cos(2\pi t)(\cos(\lambda t)+\sin(\lambda t))$ and $\lambda=0.5$, we get the trajectories, which are plotted in Fig. 1:
$X_t(\omega_1)=2\cos(2\pi t)(\cos(\lambda t)-\sin(\lambda t))$, $X_t(\omega_2)=2\cos(2\pi t)$, $X_t(\omega_3)=-2\cos(2\pi t)$, and $X_t(\omega_4)=2\cos(2\pi t)(\cos(\lambda t)+\sin(\lambda t))$.

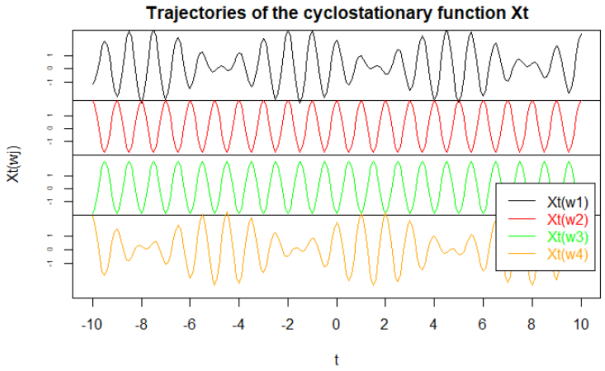

Fig. 1: A particular case, where $m=4$, of trajectories of $X_t(\omega_1)$, $X_t(\omega_2)$, $X_t(\omega_3)$ and $X_t(\omega_4)$, $t\in[-10;10]$

.

As for the $L^2([0;1[)$-stationary series, we get

$(\mathcal{I}X'_n)(\omega_1):t\mapsto 2\cos(\lambda n)f_1(t)-2\sin(\lambda n))f_4(t)=2\cos(2\pi t)(\cos(\lambda(n+t))-\sin(\lambda(n+t))$,

$(\mathcal{I}X'_n)(\omega_2):t\mapsto f_2(t)=2\cos(2\pi t)$, $(\mathcal{I}X'_n)(\omega_3):t\mapsto f_3(t)=-2\cos(2\pi t)$,

$(\mathcal{I}X'_n)(\omega_4):t\mapsto 2\sin(\lambda n)f_2(t)-2\sin(\lambda n))f_4(t)=2\cos(2\pi t)(\cos(\lambda(n+t))+\sin(\lambda(n+t))$.

The spectral components are then $Z_1=\frac{1}{2}(y_1+iy_4)(f_1-if_4)$, $Z_2=(y_2-y_3)f_2$ and $Z_3=\frac{1}{2}(y_1-iy_4)(f_1+if_4)$. Note that $\mathcal{I}X'_n=\mathrm{e}^{-i\lambda n}Z_1+Z_2+\mathrm{e}^{i\lambda n}Z_3$. Their respective first principal components are $C_1(Z_1)=\frac{1}{2}\|f_1-if_4\|(y_1+iy_4)$, $C_1(Z_2)=\|f_2\|(y_2-y_3)$, and $C_1(Z_3)=\frac{1}{2}\|f_1+if_4\|(y_1-iy_4)$, all elements of $L^2(\Omega)$.

The unidimensional stationary series $(u_n)_{n\in\mathbb{Z}}$, resulting from the PCA in the frequency domain of $(\mathcal{I}X'_n)_{n\in\mathbb{Z}}$ is then such that, for any n from $\mathbb{Z}$, $u_n=\mathrm{e}^{-i\lambda n}\frac{1}{2}\|f_1-if_4\|(y_1+iy_4)+\|f_2\|(y_2-y_3)+\mathrm{e}^{i\lambda n}\frac{1}{2}\|f_1+if_4\|(y_1-iy_4)=\|f_1+if_4\|(\cos(\lambda n)y_1+\sin(\lambda n)y_4)+\|f_2\|(y_2-y_4)$.

So we have $u_n(\omega_1)=2\|f_1+if_4\|\cos(\lambda n)$, $u_n(\omega_2)=2\|f_2\|$, $u_n(\omega_3)=-2\|f_2\|$, and finaly, $u_n(\omega_4)=2\|f_1+if_4\|\sin(\lambda n)$ (plotted on Fig. 2). These constitute the first principal component, which is composed of four trajectories in $\mathbb{Z}$ (because

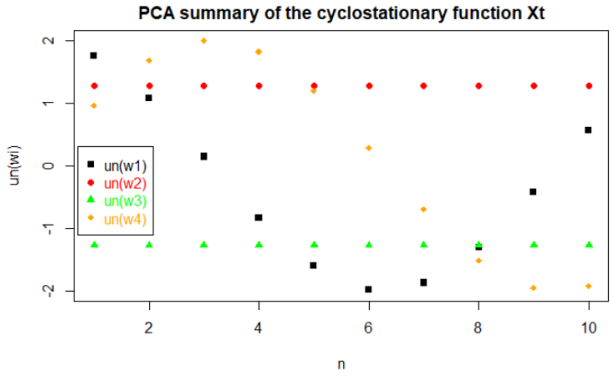

Fig. 2: The four trajectories of the unidimensional series $(u_n)_{n\in\mathbb{Z}}$, resulting from the PCA of the $L^2(\Omega)-$stationary series $(\mathcal{I}X'_n)_{n\in\mathbb{Z}}$, deduced from the cyclostationary r.f. $(X_t)_{t\in\mathbb{R}}$

.

$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$). Therefore, the $\mathbb{C}-$Hilbert space $\mathbb{C}$, of dimension 1, substitutes to the separable $\mathbb{C}-$Hilbert space $L^2([0;1[)$, of infinite dimension.

As for the reconstructed cyclostationary function with the first step of the PCA (see Section 7.4), it is the same as the initial data for this example, all the function is reconstructed with only one step. All the information included in $\{X_t; t \in \mathbb{R}\}$ is contained in $\{u_n; n \in \mathbb{Z}\}$.

9. An hypothesis of independence

In this last section, we introduce an hypothesis of independence, which places us in the particular case, mentioned at the end of Section 2.6, where the PCA in the frequency domain and the classic PCA are equivalent. We consider the particular case where $k = 1$ and $\Delta = 1$ (this implies that $\Lambda = [0;1[$ and that η is a bounded measure defined on ξ, $\sigma-$field trace of $\mathcal{B}_{\mathbb{R}}$ on Λ).

We consider two independent sub-$\sigma-$fields from $\mathcal{A}$, $\mathcal{B}_1$ and $\mathcal{B}_2$, such that the family $\{A_1 \cap A_2; (A_1, A_2) \in \mathcal{B}_1 \times \mathcal{B}_2\}$ generates $\mathcal{A}$. As the family $\{A_1 \cap A_2; (A_1, A_2) \in \mathcal{B}_1 \times \mathcal{B}_2\}$ contains Ω, is stable by finite intersection, and generates $\mathcal{A}$, we can state that $\overline{\text{vect}}\{1_{A_1\cap A_2}; (A_1, A_2) \in \mathcal{B}_1 \times \mathcal{B}_2\} = L^2(\Omega, \mathcal{A}, P)$. Let us start by some preliminary results which will be useful in the following.

Lemma 1. *For any (y, X) from $L^2(\Omega, \mathcal{B}_2, P) \times L^2_{L^2(\Lambda)}(\Omega, \mathcal{B}_1, P)$, the mapping $yX : \omega \in \Omega \mapsto y(\omega)X(\omega) \in L^2(\Lambda)$ is $\mathcal{A}-$measurable and of $P-$integrable square norm.*

Proof: For any h from $L^2(\Lambda)$, we can write $\langle y(.)X(.), h\rangle = y(.)\langle X(.), h\rangle$. As y is $\mathcal{B}_2-$measurable, it is $\mathcal{A}-$measurable. As X is $\mathcal{B}_1-$measurable, it is the same for $\langle X(.), h\rangle$, which is also $\mathcal{A}-$measurable. So the product $y(.)\langle X(.), h\rangle$ is $\mathcal{A}-$measurable, and so it is for $\langle y(.)X(.), h\rangle$.

As this is exact for any h from $L^2(\Lambda)$, which is separable, we deduce the $\mathcal{A}-$measurability of yX. Moreover, the r.m.'s $|y(.)|^2$ and $\|X(.)\|^2$ being respectively $\mathcal{B}_2-$measurable and $\mathcal{B}_1-$measurable, they are independent. So

$$\int \|y(.)X(.)\|^2_{L^2(\Lambda)} dP = \int |y(.)|^2 \|X(.)\|^2_{L^2(\Lambda)} dP = (\int |y(.)|^2 dP)(\int \|X(.)\|^2 dP) < +\infty.$$

□

If we replace $L^2(\Lambda)$ by $\mathbb{C}$, we get the following property.

Lemma 2. *For any (u_1, u_2) from $L^2(\Omega, \mathcal{B}_1, P) \times L^2(\Omega, \mathcal{B}_2, P)$, the mapping $u_1u_2 : \omega \in \Omega \mapsto u_1(\omega)u_2(\omega) \in \mathbb{C}$ is $\mathcal{A}-$measurable and of $P-$integrable square module.*

The reader will notice the consequence of the independence of the sub-$\sigma-$fields $\mathcal{B}_1$ and $\mathcal{B}_2$. For any pair $((u_1, u_2), (v_1, v_2))$ of elements from $L^2(\Omega, \mathcal{B}_1, P) \times L^2(\Omega, \mathcal{B}_2, P)$, we have

$$\langle u_1u_2, v_1v_2\rangle_{L^2(\Omega)} = \langle u_1, v_1\rangle_{L^2(\Omega,\mathcal{B}_1,P)}\langle u_2, v_2\rangle_{L^2(\Omega,\mathcal{B}_2,P)} \text{ and } \|u_1u_2\|_{L^2(\Omega)} = \|u_1\|_{L^2(\Omega,\mathcal{B}_1,P)}\|u_2\|_{L^2(\Omega,\mathcal{B}_2,P)}.$$

Let us now examine some properties of continuity.

Lemma 3. *If y is an element from $L^2(\Omega, \mathcal{B}_2, P)$, the mapping $x \in L^2(\Omega, \mathcal{B}_1, P) \mapsto xy \in L^2(\Omega)$ is continuous.*

Proof: This results from $\|xy - x'y\|_{L^2(\Omega)} = \|x - x'\|_{L^2(\Omega,\mathcal{B}_1,P)}\|y\|_{L^2(\Omega,\mathcal{B}_2,P)}$. □

In a same way, exchanging the roles of $L^2(\Omega, \mathcal{B}_1, P)$ and $L^2(\Omega, \mathcal{B}_2, P)$, we get the following.

Lemma 4. *If x is an element from $L^2(\Omega, \mathcal{B}_1, P)$, the mapping $y \in L^2(\Omega, \mathcal{B}_2, P) \mapsto xy \in L^2(\Omega)$ is continuous.*

From now on, X stands for an element from $L^2_{L^2(\Omega,\mathcal{B}_1,P)}(\Lambda)$ and $(y_n)_{n\in\mathbb{Z}}$ for a stationary series taking values in $L^2(\Omega, \mathcal{B}_2, P)$. For any t from $\mathbb{R}$, let us set $X_t = X(t-[t])y_{[t]}$. Of course, for any (λ, n) from $\Lambda\times\mathbb{Z}$, we have $X_{\lambda+n} = X(\lambda)y_n$. The random variables $X(\lambda)$ and y_n are respectively $\mathcal{B}_1-$measurable and $\mathcal{B}_2-$measurable. So they are independent random variables, the parts associated with Λ and $\mathbb{Z}$, $X(\lambda)$ and y_n, obtained by the decomposition $t = (t - [t]) + [t]$, are then independent. It is in this way that the family $\{X_t; t \in \mathbb{R}\}$ of elements of $L^2(\Omega)$ is singular. We have a first result.

Proposition 35. *The family $\{X_t; t \in \mathbb{R}\}$ is a $\Lambda-$cyclostationary r.f.*

Proof: First of all, let us notice that, from Lemma 2, $X_t = X(t-[t])y_{[t]} \in L^2(\Omega, \mathcal{A}, P)$. The mapping $x \in L^2(\Omega, \mathcal{B}_1, P) \mapsto xy_0 \in L^2(\Omega, \mathcal{A}, P)$ is continuous (cf. Lemma 3), so measurable, and as X is measurable, the same happens for $(x \in L^2(\Omega, \mathcal{B}_1, P) \mapsto xy_0 \in L^2(\Omega, \mathcal{A}, P)) \circ X$, that is for $t \in [0; 1[\mapsto X(t)y_0 \in L^2(\Omega, \mathcal{A}, P)$, evenmore for $t \in [0; 1[\mapsto X_t \in L^2(\Omega, \mathcal{A}, P)$. In addition, $\int \|X(t)y_0\|^2 \mathrm{d}\eta(t) = \int \|X(t)\|^2\|y_0\|^2 \mathrm{d}\eta(t) < \infty$, hence Point (i) of the definition of the cyclostationarity stands.

Moreover, for any pair (t, t') of elements from $\mathbb{R}$, as $X_{t+1} = X(t - [t])y_{[t]+1}$, we have

$$\begin{aligned}\langle X_{t+1}, X_{t'+1}\rangle &= \langle X(t - [t])y_{[t]+1}, X(t' - [t'])y_{[t']+1}\rangle_{L^2(\Omega,\mathcal{A},P)} = \langle X(t - [t]), X(t' - [t'])\rangle_{L^2(\Omega,\mathcal{B}_1,P)}\langle y_{[t]+1}, y_{[t']+1}\rangle_{L^2(\Omega,\mathcal{B}_2,P)}\\ &= \langle X(t - [t]), X(t' - [t'])\rangle\langle y_{[t]}, y_{[t']}\rangle = \langle X(t - [t])y_{[t]}, X(t' - [t'])y_{[t']}\rangle_{L^2(\Omega,\mathcal{A},P)} = \langle X_t, X_{t'}\rangle.\end{aligned}$$

□

Let us now define the $L^2(\Lambda)-$stationary series deduced from the $\Lambda-$cyclostationary r.f. $\{X_t; t \in \mathbb{R}\}$. From Theorem 2, the mapping $\lambda \in \Lambda \mapsto X_{\lambda+n} \in L^2(\Omega)$, that is the mapping $\lambda \in \Lambda \mapsto X(\lambda)y_n \in L^2(\Omega)$ is a representative of an element Y_n from $L^2_{L^2(\Omega)}(\Lambda)$.

In a same way as there exists an isometry (cf. Section 4) from $L^2_{L^2(\Omega)}(\Lambda)$ on $L^2_{L^2(\Lambda)}(\Omega)$, there exists an isometry $\mathcal{H}$ from $L^2_{L^2(\Omega,\mathcal{B}_1,P)}(\Lambda)$ on $L^2_{L^2(\Lambda)}(\Omega, \mathcal{B}_1, P)$ such that $\mathcal{H}(hx) = xh$, for any (h, x) element from $L^2(\Lambda) \times L^2(\Omega, \mathcal{B}_1, P)$. Then we can show that $\gamma \circ \widetilde{X} \circ \Gamma = \widetilde{\mathcal{H}X}^*$ (formula to be examined through Proposition 23). We can now formulate the following result.

Proposition 36. *For any n from $\mathbb{Z}$, we have $\mathcal{I}Y_n = y_n(\mathcal{H}X)$.*

Proof: For any (A_1, A_2, h) from $\mathcal{B}_1 \times \mathcal{B}_2 \times L^2(\Lambda)$, we can write, on one hand:

$$\begin{aligned}\langle\widetilde{\mathcal{I}Y_n}(1_{A_1}1_{A_2}), h\rangle_{L^2(\Lambda)} &= \langle\widetilde{\mathcal{I}Y_n}, (1_{A_1}1_{A_2}) \otimes h\rangle_{\sigma_2} = \langle\mathcal{I}Y_n, (1_{A_1}1_{A_2}).h\rangle_{L^2_{L^2(\Lambda)}(\mathcal{A})} = \langle Y_n, h.(1_{A_1}1_{A_2})\rangle_{L^2_{L^2(\Omega)}(\Lambda)}\\ &= \int \langle X(\lambda)y_n, h(\lambda)1_{A_1}1_{A_2}\rangle_{L^2(\Omega)}\mathrm{d}\eta(\lambda) = \int \overline{h(\lambda)}\langle X(\lambda), 1_{A_1}\rangle_{L^2(\Omega,\mathcal{B}_1,P)}\langle y_n, 1_{A_2}\rangle_{L^2(\Omega,\mathcal{B}_2,P)}\mathrm{d}\eta(\lambda)\\ &= \langle y_n, 1_{A_2}\rangle_{L^2(\Omega,\mathcal{B}_2,P)} \int \overline{h(\lambda)}\langle X(\lambda), 1_{A_1}\rangle_{L^2(\Omega,\mathcal{B}_1,P)}\mathrm{d}\eta(\lambda) \qquad (5)\\ &= \langle y_n, 1_{A_2}\rangle_{L^2(\Omega,\mathcal{B}_2,P)}\langle \int \overline{h(\lambda)}X(\lambda)\mathrm{d}\eta(\lambda), 1_{A_1}\rangle_{L^2(\Omega,\mathcal{B}_1,P)} = \langle y_n, 1_{A_2}\rangle_{L^2(\Omega,\mathcal{B}_2,P)}\langle\widetilde{X}\Gamma h, 1_{A_1}\rangle_{L^2(\Omega,\mathcal{B}_1,P)}\\ &= \langle y_n, 1_{A_2}\rangle\langle 1_{A_1}, \gamma\widetilde{X}\Gamma h\rangle = \langle y_n, 1_{A_2}\rangle\langle 1_{A_1}, \widetilde{\mathcal{H}X}^* h\rangle\end{aligned}$$

and on another hand:

$$\begin{aligned}\langle\widetilde{y_n(\mathcal{H}X)}(1_{A_1}1_{A_2}),h\rangle_{L^2(\Lambda)} &= \langle\widetilde{y_n(\mathcal{H}X)},(1_{A_1}1_{A_2})\otimes h\rangle_{\sigma_2} = \langle y_n(\mathcal{H}X),1_{A_1}1_{A_2}h\rangle_{L^2_{L^2(\Lambda)}(\Omega)}\\ &= \int\langle y_n(\omega)(\mathcal{H}X)(\omega),1_{A_1}(\omega)1_{A_2}(\omega)h\rangle_{L^2(\Lambda)}\mathrm{d}P(\omega)\\ &= \int y_n(\omega)1_{A_1}(\omega)1_{A_2}(\omega)\langle(\mathcal{H}X)(\omega),h\rangle\mathrm{d}P(\omega) = (\int y_n1_{A_2}\mathrm{d}P)(\int 1_{A_1}(\omega)\langle(\mathcal{H}X)(\omega),h\rangle\mathrm{d}P(\omega))\\ &= \langle y_n,1_{A_2}\rangle_{L^2(\Omega)}\langle\mathcal{H}X,1_{A_1}h\rangle_{L^2_{L^2(\Lambda)}(\Omega,\mathcal{B}_1,P)} = \langle y_n,1_{A_2}\rangle\langle\widetilde{\mathcal{H}X},1_{A_1}\otimes h\rangle = \langle y_n,1_{A_2}\rangle\langle 1_{A_1},\widetilde{\mathcal{H}X}^*h\rangle.\end{aligned} \tag{6}$$

From (5) and (6), we deduce

$$\langle\widetilde{\mathcal{I}Y_n}(1_{A_1}1_{A_2}),h\rangle = \langle\widetilde{y_n(\mathcal{H}X)}(1_{A_1}1_{A_2}),h\rangle.$$

Hence, considering the property of density of the family $\{1_{A_1}1_{A_2};(A_1,A_2)\in\mathcal{B}_1\times\mathcal{B}_2\}$,

$$\widetilde{\mathcal{I}Y_n} = \widetilde{y_n(\mathcal{H}X)},$$

or evenmore

$$\mathcal{I}Y_n = y_n(\mathcal{H}X).$$

□

The $L^2(\Lambda)-$stationary series deduced from the $\Lambda-$cyclostationary r.f. is then $(y_n(\mathcal{H}X))_{n\in\mathbb{Z}}$. From Section 2.6, the PCA in the frequency domain of this last is equivalent to the PCA of each of the random vectors $y_n(\mathcal{H}X)$ and is deduced from the PCA of $\mathcal{H}X$, element from $L^2_{L^2(\Lambda)}(\Omega,\mathcal{B}_1,P)$.

This study can be generalized to other kinds of cyclostationarity, in particular cyclostationary series.

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